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MEASURABLE HOMOMORPHISMS OF LOCALLY COMPACT GROUPS

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ABSTRACT. Let G and H be locally compact groups and φ a homomorphism from G into H. Suppose that $\varphi^{-1}(U)$ is measurable for every open set $U \subset H$. It is known under some conditions, for example, if H is σ-compact, that φ is continuous. Here it is shown that this result is true without any countability restrictions on G and H. The proof depends on the observation that the regular representation of H is a homomorphism.

There are a variety of results which assert that a homomorphism between two topological groups which is measurable in one sense or another is continuous. Perhaps the oldest of these is a result of Banach [1, Chapter 1, Théorèm 4] which states that a Borel measurable homomorphism between polonaise groups is continuous. For locally compact groups it is known that a Haar measurable homomorphism into a σ-compact group is continuous [5, 22.18]. Recently, Moskowitz, making ingenious use of unitary representations, has given another proof of this [6, Theorem 1]. Here, we want to show that this result is true without any countability restrictions on the groups.

The measures referred to throughout are left Haar measures on the appropriate groups.

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Theorem 1. *Let G and H be locally compact groups and $\varphi: G \to H$ a homomorphism with the property that $\varphi^{-1}(U)$ is measurable for every open subset U of H. Then φ is continuous.*

We can relax the requirement that φ be a homomorphism, but not without requiring something additional.

Theorem 2. *Let G and H be locally compact groups and $\varphi: G \to H$ a map which satisfies:*

(1) $\varphi(x)\varphi(y) = \varphi(xy)$, locally almost all $x, y \in G \times G$, and $\varphi(x^{-1}) = \varphi(x)^{-1}$, locally almost all $x \in G$.

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(2) $\varphi^{-1}(U)$ *is measurable for every open set* $U \subset H$.

(3) *There is an open relatively compact neighborhood* V *of* $e \in H$ *so that* $\varphi^{-1}(V)$ *is not locally null.*

Then there is a continuous homomorphism $\varphi_*: G \to H$ *so that* $\varphi(x) = \varphi_*(x)$, *lae.*

For brevity, we shall say that a function f with the property that $f^{-1}(U)$ is measurable for all open U is *semimeasurable.* In general, semimeasurable functions are not measurable [2, Chapter IV, §5].

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We first prove Theorem 2 and begin the proof with the following observation, which is certainly not new.

Lemma 1. *The left regular representation* λ *of* H *is a homeomorphism into the group* $\mathscr{U}(L^2(H))$ *of unitary operators on* $L^2(H)$ *provided with the strong operator topology.*

Proof. Since λ is continuous and faithful, we must show that λ is open, and because λ is an isomorphism it is enough to show that $\lambda(V)$ is a neighborhood of 1 in $\lambda(H)$, for every neighborhood V of e in H. This is equivalent to showing that if $\{y_\alpha\}_{\alpha\in A}$ is a net in H which does not converge to e, then $\{\lambda(y_\alpha)\}_{\alpha\in A}$ is a net which does not converge to 1. But if $y_\alpha \not\to e$, there is a neighborhood V of e in H and a subnet $\{y_{\alpha(\beta)}\}_{\beta\in B}$ so that $y_{\alpha(\beta)} \notin V$, $\beta \in B$. Choose a neighborhood W of e in H so that $WW^{-1} \subset V$. Let f be the characteristic function of W. Then $(\lambda(y_{\alpha(\beta)})f, f) = 0$, all $\beta \in B$, and this implies $\lambda(y_{\alpha(\beta)}) \not\to 1$.

Now suppose φ satisfies the hypotheses of Theorem 2. Then $x \to \lambda(\varphi(x)) = \pi(x)$ is a map of G into the group of unitary operators on $L^2(H)$ with the property that for all $u, v \in L^2(H)$, $x \to (\pi(x)u, v)$ is measurable. For $f \in L^1(G)$ put $\pi(f) = \int_G \pi(x)f(x)\,dx$. The usual computations show that $f \to \pi(f)$ is a $*$ representation of $L^1(G)$. Let us show it is nondegenerate. Choose $u \in L^2(H)$, $u \neq 0$. There is an open relatively compact neighborhood U of $e \in H$, $U \subset V$ so that $h(y) = (\lambda(y)u, u) \neq 0$, $y \in U$. Choose an open neighborhood U_1 of e in H so that $U_1U_1^{-1} \subset U$. Since V is relatively compact, there are $y_1, \dots, y_n \in H$ so that $V \subset \bigcup_1^n U_1y_k$, and $\varphi^{-1}(V) \subset \bigcup_1^n \varphi^{-1}(U_1y_k)$. Since $\varphi^{-1}(V)$ has positive measure, for at least one k, $\varphi^{-1}(U_1y_k)$ has positive measure as does $\varphi^{-1}(U_1y_k)\varphi^{-1}(U_1y_k)^{-1}$. For locally almost all $x_1, x_2 \in \varphi^{-1}(U_1y_k)$,

$$\varphi(x_1x_2^{-1}) = \varphi(x_1)\varphi(x_2)^{-1} \in U_1y_k(U_1y_k)^{-1} \subset U.$$

Thus $\varphi^{-1}(U)$ contains all of $\varphi^{-1}(U_1y_k)\varphi^{-1}(U_1y_k)^{-1}$, except for a locally null set, and has positive measure. Let K be a compact subset of $\varphi^{-1}(U)$ of positive measure, and put

$$f(x) = \begin{cases} (u, \lambda(\varphi(x))u), & x \in K \\ 0, & x \notin K. \end{cases}$$

Then

$$(\pi(f)u, u) = \int_K |(\lambda(\varphi(x))u, u)|^2 \, dx > 0.$$

Thus $\pi(f)u \neq 0$. This shows π is nondegenerate. It is well known [3, 13.3.4] that there is a continuous unitary representation π_* of G so that $\pi(f) = \int \pi_*(x) f(x) \, dx$, all $f \in L^1(G)$. It follows from this that $\pi(x) = \pi_*(x)$, lae. Put $G_1 = \pi_*^{-1}(\lambda(H))$. G_1 is a subgroup of G and because $\pi = \pi_*$ lae, G_1 has a locally null complement. Such a subgroup is the whole group. We can now define φ_* by putting $\varphi_*(x) = \lambda^{-1}(\pi_*(x))$. φ_* is a continuous homomorphism which agrees with φ lae.

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To prove Theorem 1 we shall show that if φ is a semimeasurable homomorphism, there is a relatively compact neighborhood V of e in H so that $\varphi^{-1}(V)$ is not locally null. It then follows from Theorem 2 that there is a continuous homomorphism φ_* so that $\varphi_* = \varphi$ lae. But now φ and φ_* agree on a subgroup with a locally null complement, and the only such subgroup is the whole group.

For the proof we need to know the existence of certain nonmeasurable sets. We use ∞ to denote a countable infinity.

Lemma 2. *The following conditions on a subgroup N of G are equivalent*:

(1) *for all $x \in G$, $[N: xNx^{-1} \cap N] \leq \infty$,*

(2) *each double coset NxN is a union of countably many left N cosets,*

(3) *each double coset NxN is a union of countably many right N cosets,*

(4) *for each $x \in G$ there is a countable set D so that $Nx \subset DN$.*

(5) *for each $x \in G$ there is a countable set D' so that $xN \subset ND'$,*

(6) *if C is a countable subset of G and M is the subgroup generated by $N \cup C$, then $[M: N] \leq \infty$.*

Proof. The double coset NxN is the orbit under right translation by N of $Nx \in N\backslash G$. The stabilizer is $xNx^{-1} \cap N$. This shows $(1) \twoheadrightarrow (2)$. Clearly, $(2) \twoheadrightarrow (4)$ and $(3) \twoheadrightarrow (5)$. The map $x \to x^{-1}$ shows that $(4) \twoheadrightarrow (5)$. To show these imply (6) it is enough to find a subgroup $M_1 \supset M$ so that $[M_1: N] \leq \infty$. Replacing C by $C \cup C^{-1}$, we may suppose $C = C^{-1}$. Since $C = C^{-1}$, the subgroup M_1 generated by $N \cup C$ is the union of all sets of the form $Nc_1Nc_2 \cdots c_kN$. By induction on k, using (3), it can be proved that each such set is a countable union of right N cosets. Thus M_1 is a countable union of right N cosets. Finally, suppose (6) holds. Choose $x \in G$ and let M be the subgroup generated by x and N. Since M is a union of countably many N cosets, so is NxN which is contained in M.

We shall say that a subgroup with any of these properties is *asoo* (a term suggested by R. Lipsman). Of course, countable or normal subgroups are asoo. An open σ-compact subgroup is asoo. If $p: G \to H$ is a homomorphism and L is asoo in H then $p^{-1}(L)$ is asoo in G. If p is onto and N is asoo in G, then $p(N)$ is asoo in H. If N is asoo in G and C is a countable subset of G, the group generated by $N \cup C$ is asoo.

Lemma 3. *Let G be a compactly generated locally compact group and N a null asoo subgroup of G. Then there is a nonmeasurable set $S \subset G$ so that $S = NS$.*

Proof. In the following, μ denotes a left Haar measure. We begin with the case that G is separable. (It is not necessary here to assume G is compactly generated.) Let $U_1 \supset U_2 \supset \cdots$ be a basis for the neighborhoods of e. Since no finite union of N cosets contains a nonempty open set, we may choose elements $x_0, x_1, \cdots$ as follows. Put $x_0 = e$. Having chosen $x_0, x_1, \ldots, x_n$, we choose $x_{n+1} \in U_{n+1} - \bigcup_1^n x_i N$. Let M be the group generated by $N \cup \{x_1, x_2, \ldots\}$, and let Y be a set of right coset representatives of M in G. Put $S = NY$. Suppose S is measurable. If X is a set of left coset representatives of N in M then X is countable and $G = XS$. Since $\mu(G) \le \sum_{x \in X} \mu(xS) \le \infty \ \mu(S)$, $\mu(S) > 0$. Let $F \subset S$ be a set of finite positive measure. For some n, the function $x \to \mu(F \cap xF) > 0$ for all $x \in U_n$. But $F \cap x_n F \subset S \cap x_n S = \varnothing$, and this is a contradiction. Thus S is nonmeasurable.

We next consider the case that N is not closed in G. Let C be a countably infinite subset of $\overline{N} - N$, let M be the subgroup generated by $N \cup C$, let X be a set of left coset representatives of N in M, Y a set of right coset representatives of M in G. Because N is asoo, X is countable. Put $S = NY$. If $x \in X - N$, $xNS \cap S = \varnothing$. Suppose S is measurable. Because $G = XS$, S must have positive measure. Let $F \subset S$ be a set of finite positive measure. The function $x \to \mu(F \cap xF)$ is continuous and positive at e. Thus there is a neighborhood U of e so that $\mu(F \cap zF) > 0$, all $z \in U$. If $x \in X - N$, the coset xN is dense in $\overline{N}$, so that $xN \cap U \neq \varnothing$. Thus there is $y \in U \cap xN$ with $\mu(F \cap yF) > 0$, but $y = xn$ for some $n \in N$, and $F \cap yF \subset S \cap xnS = \varnothing$, which is a contradiction.

To finish the proof, we need to consider the case that N is closed. Because G is compactly generated, there is a family $\{K_\alpha\}$ of compact normal subgroups of G so that G/K_α is separable, and $G = \varprojlim G/K_\alpha$ [4, Theorem 3]. Let p_α be the canonical map of G on G/K_α. $p_\alpha(N)$ is asoo. If for some α, $p_\alpha(N)$ is null, then by the first part of the proof there is a nonmeasurable set $S_1 \subset G/K$ so that $S_1 = p_\alpha(N)S$. $S = p_\alpha^{-1}(S_1)$ is the desired nonmeasurable set. What remains is the case that N is closed and there is a compact normal subgroup K such that NK is open. Then NK/N is infinite. Let C be a countable subset of NK whose image in NK/N is infinite, and let M be the subgroup of NK generated by $N \cap C$. M is asoo. If it is closed in NK, M/N is closed in NK/N, and because $[M : N] \le \infty$, N is open in M. Thus M/N is a discrete, closed subset of the compact space NK/N, thus finite. This contradicts the choice of C. Thus M is not closed, and by the second part of the proof there is a nonmeasurable set $S \subset NK$ so that $S \subset NS \subset MS = S$. Since NK is open in G, S is nonmeasurable in G. This completes the proof.

With a little bit more work, one can show that if N is a null subgroup of G and there is a compactly generated open subgroup $G_0 \subset G$ so that $N \cap G_0$ is asoo in G_0, for example, if N is normal in G, then there is a nonmeasurable subset $S \subset G$ so that $NS = S$.

We can now complete the proof of Theorem 1. Let $\varphi: G \to H$ be a semimeasurable homomorphism. To show φ is continuous it is enough to show that the restriction of φ to any open subgroup is continuous. Thus we may replace G by an open compactly generated subgroup and assume that G is compactly generated. Suppose there is an open relatively compact neighborhood V of e in H so that $\varphi^{-1}(V)$ is null. Let L be an open σ-compact subgroup of H. Since L is contained in the union of countably many translates of V, $N = \varphi^{-1}(L)$ is contained in the union of countably many translates of $\varphi^{-1}(V)$, and is null. Because L is asoo in H, N is asoo in G. By Lemma 3, we know there is a nonmeasurable subset $S \subset G$ so that $NS = S$. Let $T = \varphi(S)$. LT is open in H and by hypothesis, $\varphi^{-1}(LT)$ is measurable. But $S \subset \varphi^{-1}(LT) = \varphi^{-1}(L)\varphi^{-1}(T) \subset N(NS) = S$. This is a contradiction. Thus $\varphi^{-1}(V)$ is not null. We now know from the remarks at the beginning of this section that φ is continuous.

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