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OPERATOR ALGEBRAS AND THE CONJUGACY OF TRANSFORMATIONS II

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ABSTRACT. We prove that two automorphisms of L^∞-spaces are conjugate if and only if certain related operator algebras are algebraically isomorphic. This extends a result of W. Arveson by dropping the assumptions that the automorphisms are ergodic and measure-preserving.

W. B. Arveson [1] first looked at the conjugacy problem for automorphisms of measure spaces in terms of operator algebras. Suppose (X, μ) is a finite measure space and α is an ergodic measure-preserving automorphism of $L^\infty(\mu)$. Arveson [1] showed that the Banach algebra of operators on $L^2(X, \mu)$ generated by the unitary operator U_α of composition with α, together with the multiplications by $L^\infty(X, \mu)$-functions, classifies α in the sense that two such automorphisms are conjugate if and only if the associated algebras are unitarily equivalent.

Later, Arveson and K. B. Josephson [2] extended this result by showing that the algebras need only be isomorphic. This was proved not in the setting of measure-preserving automorphisms but of homeomorphisms on locally compact spaces. The Arveson-Josephson result was later extended by J. Peters [4] using semi-crossed products.

In [3] the present authors associated with each homeomorphism on a compact Hausdorff space a family of algebras called conjugacy algebras. It was then proved that two homeomorphisms are conjugate if and only if some conjugacy algebra for the first is isomorphic to some conjugacy algebra for the second. The algebras considered in [1, 2], and [4] are conjugacy algebras, so the results in [3] include these.

In this note, we return to the automorphisms of Arveson's original paper [1]. We show that two such automorphisms are conjugate if the corresponding algebras are isomorphic, and we do not need to assume that the automorphisms

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are either ergodic or measure-preserving. Although our results hold for L^p-spaces, $1 \le p \le \infty$, for simplicity, we stick with $p = 2$.

Suppose (X, μ) and (Y, ν) are finite measure spaces and $\tau: L^\infty(\mu) \to L^\infty(\nu)$ is an (algebra) isomorphism. Let $\mathfrak{M}(\mu)$ denote the *measure algebra* of μ, i.e., the σ-algebra of μ-measurable sets modulo the sets of measure zero. Then τ induces a lattice isomorphism between $\mathfrak{M}(\mu)$ and $\mathfrak{M}(\nu)$ defined by $\chi_{\tau(E)} = \tau(\chi_E)$. The measure $\nu\tau$ defined by $(\nu\tau)(E) = \nu(\tau(E))$ is mutually absolutely continuous with respect to μ, i.e., $(\nu\tau)(E) = 0$ if and only if $\tau(\chi_E) = 0$ if and only if $\chi_E = 0$ if and only if $\mu(E) = 0$. Let $g_\tau = (d\mu\tau^{-1}/d\nu)^{1/2}$.

The map τ also extends to an isomorphism from the algebra of all μ-measurable functions to the algebra of all ν-measurable functions (identifying functions that agree almost everywhere). The map $U_\tau: L^2(\mu) \to L^2(\nu)$ defined by $U_\tau f = g_\tau \tau(f)$ is a unitary operator.

Suppose $\alpha: L^\infty(\mu) \to L^\infty(\mu)$ is an automorphism. We let $\mathfrak{G}(\alpha)$ be the norm closed algebra of operators on $L^2(\mu)$ generated by the unitary operator U_α and the multiplication operators M_φ with $\varphi \in L^\infty(\mu)$ defined by $M_\varphi f = \varphi f$. If β is an automorphism of $L^\infty(\nu)$, we say that α and β are *conjugate* if there is an isomorphism $\tau: L^\infty(\mu) \to L^\infty(\nu)$ such that $\tau \circ \alpha = \beta \circ \tau$. In this case it is clear that $U_\tau M_\varphi U_\tau^{-1} = M_{\tau(\varphi)}$ for each φ in $L^\infty(\mu)$ and $U_\tau U_\alpha U_\tau^{-1} = M_h U_\beta$ for some unitary h in $L^\infty(\nu)$. Thus the conjugacy of α and β implies the unitary equivalence of $\mathfrak{G}(\alpha)$ and $\mathfrak{G}(\beta)$. The converse of this result when α and β are ergodic and measure-preserving is the aforementioned result of Arveson [1].

Theorem. *Suppose (X, μ) and (Y, ν) are finite measure spaces and $\alpha: L^\infty(m) \to L^\infty(\mu)$, $\beta: L^\infty(\nu) \to L^\infty(\nu)$ are automorphisms. If $\mathfrak{G}(\alpha)$ and $\mathfrak{G}(\beta)$ are algebraically isomorphic, then α and β are conjugate.*

The proof is obtained by decomposing the algebra $\mathfrak{G}(\alpha)$ as a direct product of algebras $\mathfrak{G}(\alpha_n)$, $1 \le n \le \infty$, with α_n n-periodic when $n < \infty$, and α_∞ aperiodic (freely acting). We then reduce the proof to the cases where α and β are both n-periodic (which is easy) and the aperiodic case (which is contained in [3] or [4]).

Recall that $L^\infty(\mu)$ is a maximal Abelian self-adjoint algebra of operators on $L^2(\mu)$, i.e., the only operators that commute with all the multiplication operators are the multiplication operators. Thus the center of $\mathfrak{G}(\alpha)$ consists of the multiplication operators M_φ with $\alpha(\varphi) = \varphi$. Hence the central projections in $\mathfrak{G}(\alpha)$ correspond to measurable sets E with $\alpha(E) = E$. a.e. (μ).

We order the μ-measurable sets by the relation $<$ defined so that $E < F$ means that $E \subset F$ and $\mu(E) < \mu(F)$. Since μ is finite, any chain of measurable set is order-isomorphic to a subset of the real line, where $E \to \mu(E)$ defines the order-isomorphism. Thus any chain of measurable sets has a countable cofinal subset. It is this fact that allows us to apply Zorn's lemma to obtain maximal elements.

If n is a positive integer, we say that α is *n-periodic* if there is a measurable set E such that $\{\alpha^k(E)\colon 0 \le k \le n-1\}$ is disjoint, $\bigcup_k \alpha^k(E) = X$, and $\alpha^n(f\chi_E) = f\chi_E$ for every f in $L^\infty(\mu)$. In this case it is clear that α is conjugate to the measurable automorphism $\hat{\alpha}$ on the direct sum of n copies of $L^\infty(\mu|E)$ defined by $\hat{\alpha}(f_1, \dots, f_n) = (f_n, f_1, \dots, f_{n-1})$.

We say that α is *aperiodic* or *freely acting* if, for each set E with $\mu(E) > 0$ and each positive integer n, there is an F contained in E with $\mu(F) > 0$ such that $\{\alpha^k(F)\colon 0 \le k \le n\}$ is disjoint.

It is clear that the product of countably many L^∞-spaces is an L^∞-space, i.e., $\prod L^\infty(\mu_k) = L^\infty(\sum^\oplus \mu_k)$ where $\mu = \sum^\oplus \mu_k$ is defined on the disjoint union of the X_κ's by $\mu(E) = \sum_k \mu_k(E \cap X_k)/2^k(1 + \mu_k(X_\kappa))$.

Lemma 1. *If (X, μ) is a finite measure space and α is an automorphism of $L^\infty(\mu)$, then α can be written as a direct sum $\alpha = \sum^\oplus_{1 \le n \le \infty} \alpha_n$ such that α_∞ is aperiodic and α_n is n-periodic for $1 \le n < \infty$.*

Proof. Choose a measurable set E_1 maximal with respect to the property $\alpha(f\chi_{E_1}) = f\chi_{E_1}$ for every f in $L^\infty(\mu)$. For each positive integer n, define E_{n+1} inductively to be maximal with respect to the properties $\alpha^j(E_k) \cap \alpha^i(E_{n+1}) = \varnothing$ whenever $k < n+1$ or whenever $k = n+1$ and $j \ne i \bmod (n+1)$, and $\alpha^{n+1}(f\chi_{E_{n+1}}) = f\chi_{E_{n+1}}$ for every f in $L^\infty(\mu)$. Let X_∞ be the complement in X of $\bigcup_n \bigcup_j \alpha^j(E_n)$. It is clear that we need only show that the restriction of α to $L^\infty(\mu|X_\infty)$ is aperiodic.

Suppose $E \subset X_\infty$ and $\mu(E) > 0$. It follows from the maximality of E_1 that there is an $A_1 \subset E$ such that $\alpha(\chi_A) \ne \chi_A$. Thus either $\mu(\alpha(A) \setminus A) > 0$ or $\mu(A \setminus \alpha(A)) > 0$; in the second case let $F_1 = A \setminus \alpha(A)$ and in the first case let $F_1 = A \setminus \alpha^{-1}(A)$. Clearly, $\mu(F_1) > 0$ and $F_1 \cap \alpha(F_1) = \varnothing$. It now follows from the maximality of E_2 that there is a $B \subset F_1$ with $\mu(B) > 0$ such that $\alpha^2(\chi_B) \ne \chi_B$. We let F_2 be $B \setminus \alpha^2(B)$ or $B \setminus \alpha^{-2}(B)$ depending on which has positive measure. Then $\{F_2, \alpha(F_2), \alpha^2(F_2)\}$ is disjoint and $\mu(F_2) > 0$. The proof is completed by the obvious induction. □

The E_n's in the preceding theorem are not unique (except when $n = 1$). To see this, suppose E_n is a disjoint union of measurable sets $A_0, A_1, \cdots, A_{n-1}$, and replace E_n with $\bigcup_k \alpha^k(A_k)$. However, the sets X_∞ and $X_n = \bigcup_k \alpha^k(E_n)$ $(1 \le n < \infty)$ are unique. Moreover, if we let P_n denote multiplication by the characteristic function of X_n, then we can define the P_n's algebraically in terms of $\mathfrak{G}(\alpha)$. Throughout, $\mathfrak{M}_n$ denotes the set $n \times n$ complex matrices, and $\mathfrak{M}_n(C(K))$ denotes the $n \times n$ matrices over $C(K)$.

Lemma 2. *For each positive integer n, the projection P_n is the maximal central projection in $\mathfrak{G}(\alpha)$ with the property that $P_n\mathfrak{G}(\alpha)$ is isomorphic to $\mathfrak{M}_n(C(K))$ for some compact Hausdorff space K.*

Proof. It is clear that $P_n\mathfrak{G}(\alpha)$ is isomorphic to $\mathfrak{M}_n(L^\infty(\mu|E_n))$ for $1 \le k \le \infty$.

Suppose P is a central projection in $\mathfrak{G}(\alpha)$ such that $P\mathfrak{G}(\alpha)$ is isomorphic to $\mathfrak{M}_n(C(K))$ for some compact Hausdorff space K. If $\rho\colon \mathfrak{G}(\alpha) \to \mathfrak{M}_k$ is a surjective homomorphism and $k \neq n$, then $\rho(P) = 0$, since $\rho(P\mathfrak{G}(\alpha))$ is an ideal in $\mathfrak{M}_k$ and there is no nonzero homomorphism from $\mathfrak{M}_n(C(K))$ onto $\mathfrak{M}_k$. Thus $PP_k = 0$ for $1 \leq k < \infty$, $k \neq n$.

On the other hand, suppose $\pi\colon \mathfrak{G}(\alpha) \to \mathfrak{M}_n$ is a surjective homomorphism. It follows from $U_\alpha M_\varphi = M_{\alpha(\varphi)} U_\alpha$ for every φ in $L^\infty(\mu)$ that $\ker \pi(U_\alpha)$ is an invariant subspace for $\pi(\mathfrak{G}(\alpha))$. Hence $\pi(U_\alpha)$ is invertible in $\mathfrak{M}_n$. Suppose via contradiction that $PP_\infty \neq 0$. The aperiodicity of α_∞ implies that there is a nonzero projection q in $L^\infty(\mu)$ such that $\{\alpha^{-k}(q)\colon 1 \leq k \leq n^2 + 1\}$ is an orthogonal family of projections. Since $\dim \mathfrak{M}_n = n^2$, there is a k, $1 \leq k \leq n^2 + 1$, such that $\pi(M_{\alpha^{-k}(q)}) = 0$. It follows from $U_\alpha^k M_{\alpha^{-k}(q)} = M_q U_\alpha^k$ that $\pi(M_q)\pi(U_\alpha^k) = 0$. Since $\pi(U_\alpha)$ is invertible, we conclude that $\pi(M_q) = 0$. Since π was arbitrary and M_q is a nonzero element of $P\mathfrak{G}(\alpha)$, we contradict the fact that $P\mathfrak{G}(\alpha)$ is isomorphic to $M_n(C(K))$. Hence $PP_\infty = 0$.

Since $PP_k = 0$ for $1 \leq k \leq \infty$, $k \neq n$, it follows that $P \leq P_n$. This proves the maximality of P_n. □

Proof of the theorem. It follows from Lemma 1 and Lemma 2 that $\mathfrak{G}(\alpha)$ is isomorphic to $\mathfrak{G}(\beta)$ implies that $\mathfrak{G}(\alpha_n)$ is isomorphic to $\mathfrak{G}(\beta_n)$ for $1 \leq n \leq \infty$. It is clear that if α_n and β_n are conjugate for $1 \leq n \leq \infty$, then α and β are conjugate. This reduces the problem to the cases in which α and β are either both aperiodic or both n-periodic for some n, $1 \leq n < \infty$.

If α is n-periodic, it is clear, via conjugacy, that we can assume that $L^\infty(\mu) = L^\infty(\mu_0) \oplus \cdots \oplus L^\infty(\mu_0)$ (n summands) and $\alpha(f_1, f_2, \dots, f_n) = (f_n, f_1, \dots, f_{n-1})$. In this case $\mathfrak{G}(\alpha)$ is isomorphic to $\mathfrak{M}_n(L^\infty(\mu_0))$. If β is also n-periodic and $L^\infty(\nu) = L^\infty(\nu_0) \oplus \cdots \oplus L^\infty(\nu_0)$ (n summands) and $\beta(f_1, f_2, \dots, f_n) = (f_n, f_1, \dots, f_{n-1})$, then $\mathfrak{G}(\alpha)$ and $\mathfrak{G}(\beta)$ being isomorphic implies that their centers $L^\infty(\mu_0)$ and $L^\infty(\nu_0)$ are isomorphic. The latter clearly implies that α and β are conjugate.

Next suppose that α and β are both aperiodic. Then $\mathfrak{G}(\alpha)$ is isomorphic to the semi-crossed product of $L^\infty(\mu)$ with respect to the automorphism α, and $\mathfrak{G}(\beta)$ is isomorphic to the semi-crossed product of $L^\infty(\nu)$ with respect to the automorphism β [1]. It follows from either [4] or [3] that α and β are conjugate.

Remarks. 1. If α is an automorphism of $L^\infty(\mu)$, we define $g_p = g_{\alpha,p}$ to be $(d\mu\alpha^{-1}/d\mu)^{1/p}$ if $1 \leq p < \infty$ and to be 1 if $p = \infty$. Then $U_{\alpha,p}(f) = g_p\alpha(f)$ defines an invertible isometry on $L^p(\mu)$. Let $\mathfrak{G}_p(\alpha)$ be the norm closed algebra of operators on $L^p(\mu)$ generated by $U_{\alpha,p}$ and the multiplications by L^∞-functions. Our proof actually shows that if $\mathfrak{G}_p(\alpha)$ is isomorphic to $\mathfrak{G}_q(\beta)$ for some p and q, then α and β are conjugate.

2. Our theorem remains true if $\mathfrak{G}(\alpha)$ is replaced by any subalgebra that contains U_α and all the multiplication operators. □

It is possible to state our theorem in an operator-theoretic context. Suppose that H is a separable Hilbert space and $\mathfrak{M}$ is a masa (i.e., maximal Abelian self-adjoint algebra of operators) on H, and U is a unitary operator on H such that $U^*\mathfrak{M}U = \{U^*TU: T \in \mathfrak{M}\} = \mathfrak{M}$. Via the spectral theorem there is a unitary equivalence between H and $L^2(\mu)$ for some finite measure μ that sends $\mathfrak{M}$ to the aglebra of all multiplications by L^∞-functions and sends U to $M_h U_\alpha$ for some unitary h in $L^\infty(\mu)$ and some automorphism α of $L^\infty(\mu)$. Define $\mathfrak{G}(\mathfrak{M}, U, H)$ to be the norm closed algebra of operators on H generated by $\mathfrak{M}$ and U.

Corollary. *Suppose $\mathfrak{M}_i$ is a masa on the separable Hilbert space H_i and U_i is a unitary operator such that $U_i^*\mathfrak{M}_i U_i = \mathfrak{M}_i$ for $i = 1, 2$. Then $\mathfrak{G}(\mathfrak{M}_1, U_1, H_1)$ is algebraically isomorphic to $\mathfrak{G}(\mathfrak{M}_2, U_2, H_2)$ if and only if there is a unitary operator $U: H_2 \to H_1$ such that $U^*\mathfrak{M}_1 U = \mathfrak{M}_2$ and $U^* U_1 U U_2^* \in \mathfrak{M}_2$.*

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