NONCOMPACT COMMUTATORS IN THE COMMUTANT OF A CYCLIC OPERATOR

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Abstract. We show that the commutant of the operator $S \oplus (I + S^*)$, where $S$ is the shift operator, contains two operators $A$ and $B$ such that $AB - BA$ is not a compact operator.

Let $S$ be the shift operator and let $T = S \oplus (I + S^*)$. Herrero [1] has shown that $T$ and $T^*$ are cyclic and the commutant of $T$ is not commutative. In this paper we show that the commutant of $T$ fails to be commutative in the following stronger sense.

Theorem. The commutant of the operator $S \oplus (I + S^*)$ contains two operators $M$, $M_1$ such that $MM_1 - M_1M$ is not a compact operator.

Proof. For convenience, let $W = S \oplus (I + S^*)$. So, $W$ is an operator (= bounded linear operator) on the Hilbert space $H = H^2 \oplus H^2$. Each operator $A$ on $H$ can be expressed as a matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where each $A_{ij} \in \mathcal{L}(H^2)$, the bounded linear operators on $H^2$. In particular,

$$W = \begin{pmatrix} S & 0 \\ 0 & I + S^* \end{pmatrix}.$$

An easy computation shows that the operators

$$E_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}$$

are in the commutant of $W$ provided that $X = C$ satisfies the operator equation

$$(I + S^*)X = XS.$$

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Furthermore, a simple computation shows that
\[ E_1E_2 - E_2E_1 = \begin{pmatrix} 0 & 0 \\ -C & 0 \end{pmatrix}. \]
These observations show that the theorem follows from Lemma 1.

Lemma 1. The operator equation (1) has a solution \( X \) in \( \mathcal{L}(H^2) \) which is not a compact operator.

To complete the proof of the theorem we now prove Lemma 1 based on the following lemmas and propositions. Let \( T \) and \( \varphi \) be as in Lemma 4. Let \( A \) be the operator on \( H^2 \) with matrix \( (a_{k,n}) \) corresponding to \( \varphi \) as in Proposition 3. Then Proposition 3 and Lemma 4 show that \( X = A \) is a solution (in \( \mathcal{L}(H^2) \)) of (1) which is not compact.

Proposition 2. The operator \( X = A \) is a solution in \( \mathcal{L}(H^2) \) of (1) if and only if the matrix \( (a_{k,n}, \ k, \ n = 0,1,2, \ldots) \), of \( A \) relative to the orthonormal basis \( u_n = e^{in\theta}, \ n = 0,1, \ldots \), satisfies:

(i) \( a_{k,n} + a_{k+1,n} = a_{k,n+1} \);

(ii) the matrix \( (a_{k,n}) \) determines a bounded operator on \( l^2 \).

Proof. Suppose \( A \) is a solution of (1) and \( (a_{k,n}) \) is its matrix. Then (ii) is clearly satisfied. To prove (i) we note
\[
\sum_{k} a_{k,n+1} u_k = Au_{n+1} = ASu_n = (I + S^*)Au_n = (I + S^*) \sum_{k} a_{k,n} u_k = \sum_{k} a_{k,n} u_k + \sum_{k} a_{k+1,n} u_k.
\]
The converse follows from similar reasoning.

Let \( L^2 \) denote \( L^2(\mu) \) where \( \mu \) is normalized Lebesgue measure on the unit circle. We regard \( H^2 \) as a subspace of \( L^2 \) whenever appropriate. For each \( \varphi \in L^2 \), we define the transformation \( T_\varphi \) from \( L^2 \) into sequences by
\[
T_\varphi f(k) = \int (e^{i\theta} - 1)^k \varphi(e^{i\theta}) f(e^{i\theta}) d\theta / 2\pi,
\]
where \( f \in L^2, \ k = 0,1, \ldots \).

Proposition 3. For \( \varphi \in L^2 \), the matrix
\[
a_{k,n} \equiv \int (e^{i\theta} - 1)^k e^{in\theta} \varphi(e^{i\theta}) d\theta / 2\pi
\]
is the matrix, relative to the orthonormal basis in Proposition 2, of a solution (compact solution) \( X = A \) of (1) if and only if the restriction of \( T_\varphi \) to \( H^2 \) determines a bounded (compact) linear transformation of \( H^2 \) into \( l^2 \).

Proof. If \( (a_{k,n}) \) is the matrix of a bounded operator, then \( T_\varphi \) clearly has the stated boundedness property. If \( T_\varphi \) has the stated boundedness property, then
(a_{k,n}) is the matrix of an operator \( A \in \mathcal{L}(H^2) \). An easy computation shows that \( (a_{k,n}) \) satisfies (i) of Proposition 2. Thus, \( X = A \) is a solution of (1).

**Lemma 4.** Let \( T = T_\varphi \) where \( \varphi \) is the characteristic function of the arc \( \{ e^{i\theta} : 0 < \theta < \pi/3 \} \). Then

(i) \( T \) is a bounded linear transformation of \( L^2 \) into \( L^2 \);  
(ii) the restriction of \( T \) to \( H^2 \) is not a compact operator.

**Proof.** For \( f \in L^2 \), 

\[
Tf(k) = \int_0^{\pi/3} (e^{i\theta} - 1)^k f(e^{i\theta}) d\theta / 2\pi.
\]

If \( b_k = \int_0^1 x^k g(x) \, dx \), then since the Hilbert matrix is bounded,

\[
\sum_k |b_k|^2 \leq C \int_0^1 |g(x)|^2 \, dx
\]

for some constant \( C \). From this and a routine change of variable argument with \( x = |e^{i\theta} - 1| \), we conclude that \( T \) is bounded.

We now prove (ii). For convenience, let \( v(\theta) = e^{i\theta} - 1 \). A routine computation shows that

\[
T^*Tf(e^{i\theta}) = \int_0^{\pi/3} \frac{1}{1 - v(\theta)} |f(\theta)|^2 d\theta / 2\pi.
\]

for \( f \in L^2 \) and \( 0 \leq \theta \leq \pi/3 \); and, \( T^*Tf(e^{i\theta}) = 0 \), otherwise. We will show that the restriction of \( T^*T \) to \( H^2 \) is not compact. For convenience, let \( M \) denote the restriction of \( T^*T \) to \( H^2 \). In particular, \( M \) is a bounded linear transformation from \( H^2 \) into \( L^2 \). We will use Lemma 5 to show that \( M \) is not compact.

**Lemma 5.** Suppose that \( H, H' \) are Hilbert spaces, that \( A \) is a bounded linear transformation from \( H \) into \( H' \) and that \( x_n \in H \) for \( n = 1, 2, \ldots \) such that

(i) \( \|x_n\| = 1 \);  
(ii) \( \langle x_m, x_n \rangle \to 0 \) as \( n \to \infty \) for each fixed \( m \);  
(iii) \( \|Ax_n\| \geq \delta > 0 \) for all \( n \).

Then \( A \) is not compact.

This lemma and Lemma 6 below show that \( M \) is not compact, which completes the proof of (ii) in Lemma 4.

For \( 0 < a < 1 \), let \( z_a = (1 + a) e^{i\pi/3} \) and \( f_a(e^{i\theta}) = \sqrt{a} / (e^{i\theta} - z_a) \). Clearly, each \( f_a \in H^2 \).

**Lemma 6.** There exist positive constants \( C_0, C_1, C_2 \) such that for \( 0 < a, d < 1 \),

(i) \( C_0 \leq \|f_a\| \leq C_1 \).
(ii) \((f_a, f_d) \to 0\) as \(d \to 0\) for each fixed \(a\), and

(iii) \(\|M f_a\| \geq C_2 \|f_a\|\) where \(\|\|\) \(, (\ , \ )\) denote the norm and inner product on \(L^2\).

**Proof.** Let \(b = 1 + a\) \((0 < a < 1)\) and let \(I = \int_{-\pi}^{\pi} 1/|e^{i\theta} - b|^2 \, d\theta\). By noting that

\[
|e^{i\theta} - b|^2 = a^2 + b\theta^2 - 2bR(\theta),
\]

where \(R(\theta) = \cos \theta - (1 - \theta^2/2)\) one can show that

\[
\left(\frac{6}{11}\right) \left(\frac{1}{a}\right) \leq I \leq \left(3\sqrt{2}\pi\right) \left(\frac{1}{a}\right);
\]

and, (i) easily follows.

To prove (ii) we note that \((f_a, f_d)\) is an integral over the interval \(-\pi < \theta < \pi\) which is the sum of integrals over the subsets \(J_1 = \{\theta: |\theta - \pi/3| < \delta\}\) and its complement \(J_2\) in \([-\pi, \pi]\). By applying the Schwarz inequality to the integrals over each interval \(J_1\) and \(J_2\), by noting that \(f_a\) \(f_d\) \(\to 0\) as \(\delta \to 0\) for a fixed and by noting that \(f_d \to 0\) as \(d \to 0\) uniformly in \(J_2\), we obtain (ii).

We now prove (iii). From the definition of \(M\),

\[
M f_a(\theta_1) = \int_0^{\pi/3} \left[1/(1 - v(\theta_1)v(\theta))\right] f_a(\theta) \, d\theta / 2\pi
\]

for \(0 < \theta_1 < \pi/3\). For convenience, let \(F = F(a, \theta_1, \theta)\) denote the integrand in the latter integral. It is easily seen that

1. the argument of \(f_a(\theta)\) varies over an interval of length \(\leq \pi/2\) for \(0 < a < 1\) and \(0 \leq \theta \leq \pi/3\).

A geometric argument shows that \(v(\theta_1)v(\theta)\) lies in the intersection of two circles both of radius one and with centers at \(e^{i\pi/3}\) and \(e^{-i\pi/3}\) for \(0 \leq \theta\), \(\theta_1 \leq \pi/3\). From this we conclude that

2. the argument of \(1 - v(\theta_1)v(\theta)\) varies over an interval of length \(\leq \pi/3\) for \(0 \leq \theta\), \(\theta_1 \leq \pi/3\); and, for some \(C > 0\),

\[
|1 - v(\theta_1)v(\theta)| \leq C(1 - |v(\theta_1)v(\theta)|), \quad 0 \leq \theta, \theta_1 \leq \pi/3.
\]

Thus, in particular, the argument of \(F\) varies over an interval of length \(\leq 5\pi/6\) for \(0 < a < 1\) and \(0 \leq \theta\), \(\theta_1 \leq \pi/3\). So, for some complex number \(\lambda\) with \(|\lambda| = 1\), the argument of \(\lambda F\) lies in the interval \([-5\pi/12, 5\pi/12]\). Therefore,

\[
|M f_a(\theta_1)| \geq C \int_0^{\pi/3} |F(a, \theta_1, \theta)| \, d\theta, \quad 0 < \theta_1 < \pi/3
\]

for some \(C > 0\).

Let \(t = |v(\theta)|\) and \(s = |v(\theta_1)|\). From (2) and the definition of \(f_a\) it follows that

\[
|F| \geq C(1/(1 - st))(1/\sqrt{a}), \quad 0 < s < 1, 1 - a < t < 1,
\]
for some \( C > 0 \). By a change of variable we conclude that for some \( C > 0 \),
\[
|M f_a(\theta_1)| \geq (C/\sqrt{a}) \int_{1-a}^{1} 1/(1 - st) \, dt \geq (C/\sqrt{a}) \log(a/2(1 - s)) \geq 0
\]
for \( s > 1 - a/2 \). Thus, for some \( C > 0 \), \( C \) independent of \( a \), we have
\[
\int_{0}^{\pi/3} |M f_a(\theta_1)|^2 \, d\theta_1 \geq (C/a) \int_{1-a/2}^{1} [\log(a/2(1 - s))]^2 \, ds = (C/2) \int_{0}^{1} (\log t)^2 \, dt,
\]
which, together with (i), proves (ii).

Acknowledgment

The referee has pointed out that the operator \( A = S^* \otimes I \) is cyclic and the commutant of \( A \) contains operators \( B \) and \( C \) such that \( BC - CB \) is not compact. However, \( A^* \) is not cyclic. So, the significance of Herrero’s example is that both \( T \) and \( T^* \) are cyclic. The referee has also noted that if \( A \) denotes the infinite direct sum of the operators \((1/n)I + 10^{-n}T\), \( n = 1, 2, \ldots \), then \( A \) and \( A^* \) are cyclic and the commutant of \( A \) contains a pair of operators with noncompact commutator. The point of our theorem is that it is not so clear that the operator \( T \) has this property. Because of the referees comments, we changed our original abstract and first paragraph.

References


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