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# ON HARMONIC BOEHMIANS

PIOTR MIKUSIŃSKI

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ABSTRACT. Existence of generalized functions (called Boehmians) satisfying the Laplace equation which are not $\mathscr{C}^\infty$-functions is proved.

By Boehmians on $\mathbf{R}^n$ we mean the space of generalized functions which can be defined as the completion of the space $\mathscr{C}^\infty(\mathbf{R}^n)$ of infinitely differentiable functions with respect to $\Delta$-convergence. A sequence of functions $f_n \in \mathscr{C}^\infty(\mathbf{R}^n)$ is called $\Delta$-*convergent* to a function $f \in \mathscr{C}^\infty(\mathbf{R}^n)$ if there exists a delta sequence $\{\delta_n\}$ such that the sequence of convolutions

$$(f_n - f) * \delta_n \tag{1}$$

converges to zero uniformly on every compact subset of $\mathbf{R}^n$. By a *delta sequence* we mean a sequence of functions $\delta_n \in \mathscr{D}(\mathbf{R}^n)$ ($\mathscr{C}^\infty$-functions with compact support) such that the following conditions are satisfied

$$\int_{\mathbf{R}^n} \delta_n = 1 \quad \text{for all } n \in \mathbf{N}, \tag{2}$$

$$\delta_n \geq 0 \quad \text{for all } n \in \mathbf{N}, \tag{3}$$

(4) For every $\varepsilon > 0$ there exists a number $M$ such that $\delta_n(x) = 0$ for all $n \geq M$ and all $x \in \mathbf{R}^n$ such that $\|x\| \geq \varepsilon$.

By the *convolution* $f * g$ of $f, g \in \mathscr{C}(\mathbf{R}^n)$ we mean the function defined by the integral $(f * g)(x) = \int_{\mathbf{R}^n} f(y)g(x-y)\,dy$. Note that in (1) the convolution is always well defined.

For other definitions and results on Boehmians see [2–10].

Since the differentiation is continuous with respect to $\Delta$-convergence, (see [5], Theorem 4.4), the derivatives of a Boehmian $F = \Delta\text{-}\lim_{n\to\infty} f_n$ can be defined as

$$\frac{\partial F}{\partial x_k} = \Delta\text{-}\lim_{n\to\infty} \frac{\partial f_n}{\partial x_k}, \qquad k = 1, \ldots, N. \tag{5}$$

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A Boehmian $F$ on $\mathbf{R}^2$ will be called *harmonic* if

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0. \tag{6}$$

It is known that every Schwartz distribution satisfying (6) is a $\mathscr{C}^\infty$-function, (see e.g. [11]). We will show that there are Boehmians satisfying the Laplace equation which are not functions.

**Theorem.** *Let $\{a_n\}$ be an arbitrary sequence of numbers and let $\{\beta_n\}$ be a sequence of positive numbers such that*

$$\sum_{n=1}^{\infty} \frac{1}{\beta_n} < \infty. \tag{7}$$

*Let*

$$f_n(x,y) = e^{\beta_n \pi x} \cos \beta_n \pi y,$$

*for $n = 1, 2, \cdots$ and $(x, y) \in \mathbf{R}^2$. Then the series*

$$\sum_{n=1}^{\infty} \alpha_n f_n \tag{8}$$

*is $\Delta$-convergent and its sum is a harmonic Boehmian.*

*Proof.* In view of Theorem 5.4 in [4], to prove $\Delta$-convergence of (8) it suffices to find a delta sequence $\{\delta_n\}$ such that for every $k \in \mathbf{N}$ the series

$$\sum_{n=1}^{\infty} \alpha_n (f_n * \delta_k)$$

converges uniformly on compact subsets of $\mathbf{R}^2$.

Let $\gamma$ be the characteristic function of the interval $[-1, 1]$. Define, for $n = 1, 2, \ldots,$ $\varphi_n(x, y) = \frac{1}{4}\beta_n^2 \gamma(\beta_n x)\gamma(\beta_n y)$. Next define

$$\delta_n = \varphi_n * \varphi_{n+1} * \cdots = \lim_{k\to\infty} (\varphi_n * \cdots * \varphi_{n+k}). \tag{9}$$

It can be proved, under condition (7), that the limit exists for every $n \in \mathbf{N}$ (uniformly on $\mathbf{R}^2$) and that the sequence $\{\delta_n\}$ is a delta sequence, (see [1 and 5]).

For fixed $k \in \mathbf{N}$ and every $n \geq k$ we have $f_n * \delta_k = 0$, because $f_n * \varphi_n = 0$, and therefore

$$\sum_{n=1}^{\infty} \alpha_n (f_n * \delta_k) = \sum_{n=1}^{k-1} \alpha_n (f_n * \delta_k),$$

which implies uniform convergence of the series. Thus (8) is $\Delta$-convergent to a Boehmian

$$F = \sum_{n=1}^{\infty} \alpha_n (f_n * \delta_k).$$

Since all $f_n$'s are harmonic functions, $F$ is a harmonic Boehmian, by (5).

Note that there are Boehmians obtained as the sum of (8) which are functions. Indeed, if $\{\alpha_n\}$ converges to zero fast enough, then the series is uniformly convergent on compact sets. To show that for some sequences $\{\alpha_n\}$ the sum of (8) is not a function we will use the fact that for every continuous function $f$ and any delta sequence $\{\delta_n\}$ we have

$$\lim_{n\to\infty} (f_n * \delta_n)(0) = f(0).$$

Since, for the delta sequence $\{\delta_n\}$ defined by (9) we have

$$(F * \delta_n)(0) = \sum_{k=0}^{n-1} \alpha_k (f_k * \delta_n)(0),$$

we can easily find a sequence $\{\alpha_n\}$ such that $(F * \delta_n)(0) \to \infty$ as $n \to \infty$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, ORLANDO, FLORIDA 32816