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# CONTINUITY PROPERTIES OF THE SPECTRUM OF OPERATORS ON LEBESGUE SPACES

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ABSTRACT. Fix $1 \leq p < s \leq \infty$. Let $T_x, x \in [p,s]$, be the collection of bounded linear operators on the Lebesgue spaces $L^x$ determined by some fixed operator $T$. This paper concerns continuity properties of the map $x \to \sigma(T_x)$.

## 1. INTRODUCTION

Let $\Omega$ be a nonempty set, and let $\mu$ be a $\sigma$-finite measure on $\Omega$. If $1 \leq p \leq \infty$, then let $L^p$ denote the usual Lebesgue space $L^p(\Omega, \mu)$ with the usual $p$-norm. Now fix $p$ and $s$ with $1 \leq p < s \leq \infty$. Let $B_{p,s}$ be the algebra of all linear operators $T: L^p \cap L^s \to L^p \cap L^s$ with the property that $T$ is continuous on $L^p \cap L^s$ with respect to both the $p$-norm and the $s$-norm. The algebra $\mathscr{B}_{p,s}$ is a Banach algebra which is studied in [2]. Also, some applications involving $\mathscr{B}_{p,s}$ can be found in [3]. If $T \in \mathscr{B}_{p,s}$, then by the Riesz Convexity Theorem [6, Theorem 11, p. 525] for each $x$ in the interval $[p,s]$ the linear operator $T$ has a unique extension to a bounded linear operator $T_x$ on $L^x$ (in the case where $s = \infty, T_\infty$ is a bounded linear operator on the closure of $L^p \cap L^\infty$ in $L^\infty$). Let $\sigma(T_x)$ and $r(T_x)$ denote the spectrum and spectral radius of this extension. It is well known that $\sigma(T_x)$ can be different for different $x$ in $[p,s]$. An example of this phenomenon is given in [4]: Let $\Omega = (0,\infty)$ and $\mu$ be the Lebesgue measure. Take $T$ to be the Cesàro operator, so

$$T_y(f)(t) = t^{-1} \int_0^t f(x)dx \qquad (f \in L^y),$$

$1 < y \leq \infty$. Then $\sigma(T_y)$ is the circle with center and radius $(2(1-y^{-1}))^{-1}$. Note that $\sigma(T_y)$ varies continuously in $y$ in this case.

In this paper we study continuity properties of the map $y \to \sigma(T_y)$. A key theorem here is that for $T \in \mathscr{B}_{p,s}$, the function $y \to r(T_y)$ is continuous on the open interval $(p,s)$ (it need not be continuous on $[p,s]$). A number of other continuity results are derived using this basic theorem as a tool.

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There are only a few examples in the literature where $\sigma(T_y)$ is specifically computed and this set is different for different values of $y$. Specific examples where this occurs can be found in [4], [5], and [7]. Next we give a number of additional examples.

**Example 1.** Let $\Omega$ be the positive integers, and let $\mu$ be the measure with

$$\mu(k) = 1/k! \qquad (k \geq 1).$$

For $1 \leq p < \infty$, let $T_p$ be the linear operator on $l^p(\mu)$ defined by $T_p(\{a_k\}) = \{b_k\}$ where $b_k = (k+1)^{-1}a_{k+1}$, $k \geq 1$. Also, let $T_\infty$ act on $c_0$ by the same rule.

*Note.* For $1 < p \leq \infty$, $T_p$ is compact and $\sigma(T_p) = \{0\}$.

*Proof.* Fix $p$, $1 < p < \infty$. Define $M_n$ on $l^p(\mu)$ by $M_n(\{a_k\}) = \{b_k\}$ where $b_k = a_k$ for $1 \leq k \leq n$, and $b_k = 0$ for $k > n$. For $\{a_k\} \in l^p(\mu)$,

$$\begin{aligned}\|(T_p - M_nT_p)\{a_k\}\|_p^p &= \sum_{k=n+1}^{\infty} (1/k!)(k+1)^{-p}|a_{k+1}|^p \\ &= \sum_{k=n+1}^{\infty} (k+1)^{1-p}(1/(k+1)!)|a_{k+1}|^p \leq (n+2)^{1-p}\|\{a_k\}\|_p^p.\end{aligned}$$

Therefore

$$\|T_p - M_nT_p\|_p \leq (n+2)^{p^{-1}-1} \to 0$$

as $n \to \infty$. This proves $T_p$ is compact. A similar computation shows that $T_\infty$ is compact.

Now it is easy to see that the operator $M_nT_p$ is nilpotent, so $\sigma(M_nT_p) = \{0\}$ for $n \geq 1$. Since $T_p$ is compact, this implies $\sigma(T_p) = \{0\}$ [8, Theorem 3].

Now we consider the case where $p = 1$. Let $l^1$ denote the usual sequence space (the $L^1$-space of the positive integers with respect to counting measure). Define an isometry $W_1 : l^1 \to l^1(\mu)$ by $W_1(\{a_k\}) = \{c_k\}$ where $c_k = k!a_k$, $k \geq 1$. The isometry $W_1$ maps $l^1$ onto $l^1(\mu)$. Let $B$ be the unilateral backward shift on $l^1$; $B(\{a_k\}) = \{b_k\}$ where $b_k = a_{k+1}$, $k \geq 1$. A straightforward computation shows

$$W_1^{-1}T_1W_1 = B \qquad \text{on } l^1.$$

As is well known, $\sigma(B) = D$, the closed unit disk, so $\sigma(T_1) = D$.

To summarize: in this case $T \in \mathscr{B}_{1,\infty}$, $T_x$ is compact with $\sigma(T_x) = \{0\}$ for $1 < x \leq \infty$, while $T_1$ is not compact with $\sigma(T_1) = D$. The spectral radius function is given by

$$r(T_x) = 0 \qquad \text{for } 1 < x \leq \infty, \qquad r(T_1) = 1.$$

**Example II.** Let $\Omega$ be the positive integers, and let $\mu$ be the measure with

$$\mu(k) = 2^k \qquad (k \geq 1).$$

Let $S$ and $B$ be defined on any sequence space by

$$S(\{a_k\}) = \{b_k\} \qquad \text{where } b_1 = 0, b_k = a_{k-1} \text{ for } k > 1;$$
$$B(\{a_k\}) = \{c_k\} \qquad \text{where } c_k = a_{k+1} \text{ for } k \geq 1.$$

Let $S_p$ and $B_p$ denote the operators $S$ and $B$ acting on the sequence space $l^p(\mu)$, $1 \leq p < \infty$. Let $l^p$ be the usual $L^p$-space on the positive integers with counting measure. For $1 \leq p < \infty$ define $W_p : l^p \to l^p(\mu)$ by $W_p(\{a_k\}) = \{d_k\}$ where $d_k = 2^{-k/p} a_k$, $k \geq 1$. Then $W_p$ is a linear isometry of $l^p$ onto $l^p(\mu)$. Each of the following properties is established by a simple computation.

(II.1) $S_2^* = 2B_2$, so $S_2 + 2B_2$ is selfadjoint on $l^2(\mu)$;

(II.2) $W_p^{-1} S_p W_p = 2^{1/p} S$ on $l^p$;

(II.3) $W_p^{-1} B_p W_p = 2^{-1/p} B$ on $l^p$.

Now the spectrum of an operator of the form $a + bS + cB$, $a, b, c \in \mathbf{C}$, on the space $l^p$ has been specifically computed by I. Gokhberg and M. Zambitsky. This result can be found in [9, Proposition 2]. Combining this information with (II.1)–(II.3), we can compute the spectrum of any operator of the form $a + bS_p + cB_p$, $1 \leq p < \infty$. We state these results for a specific operator:

Set $T_p = S_p + 2B_p \in \mathscr{B}(l^p(\mu))$, $1 \leq p < \infty$. Then

(II.4) $\sigma(T_2) = \{\lambda \in \mathbf{R} : -2\sqrt{2} \leq \lambda \leq 2\sqrt{2}\}$;

(II.5) For $p \neq 2$, $p^{-1} + q^{-1} = 1 (q^{-1} = 0$ when $p = 1)$, $\sigma(T_p)$ is the set of all $\lambda = x + iy$, $x$, $y \in \mathbf{R}$, which lie inside or on the ellipse $x^2/a^2 + y^2/b^2 = 1$ where $a = (2^{1/p} + 2^{1/q})$ and $b = (2^{1/p} - 2^{1/q})$.

(II.6) $r(T_p) = 2^{1/p} + 2^{1-1/p}$, $1 \leq p < \infty$.

The operator $T_2$ in this example is selfadjoint. In this case it is always true that when $2 \leq x \leq y$ or $y \leq x \leq 2$, then $\sigma(T_x) \subseteq \sigma(T_y)$; see [2, Theorem 4.4]. Also, the fact that $\sigma(T_p) \neq \sigma(T_2)$ for $p \neq 2$ shows that the hypothesis (4.1) in §4 of [1] cannot be omitted and still obtain the results in [1, Theorem 4.8].

## 2. Continuity properties

The main result of this paper concerns the continuity of the spectral radius function $x \to r(T_x)$; this result is stated in Theorem 3. On the way to proving this theorem, we derive some additional interesting continuity properties of the spectrum.

**Proposition 1.** *Assume $T \in \mathscr{B}_{p,s}$ and $r(T_y) = 0$ for some $y \in [p, s]$. Then $r(T_x) = 0$ for all $x \in (p, s)$.*

*Proof.* Fix $x \in (p, s)$. We may assume $x < y$. Choose $z$ such that $z \in (p, s)$ and $z < x$. Then $\exists t \in (0, 1)$ with $x^{-1} = tz^{-1} + (1 - t)y^{-1}$. Applying the Riesz Convexity Theorem, we have for all $n \geq 1$

$$\|(T^n)_x\| \leq \|(T^n)_z\|^t \|(T^n)_y\|^{1-t}.$$

Therefore

$$r(T_x) \leq r(T_z)^t r(T_y)^{1-t} = 0.$$

**Corollary 2.** *Assume $T \in \mathscr{B}_{p,s}$ and for some $y \in [p,s]$ $\sigma(T_y)$ is finite. Then $\sigma(T_x) = \sigma(T_y)$ for all $x \in (p,s)$.*

*Proof.* Assume $\sigma(T_y) = \{\lambda_1, \dots, \lambda_n\}$. Let $q$ be the polynomial $q(z) = \prod_{k=1}^n (z - \lambda_k)$. Then $\sigma(q(T_y)) = q(\sigma(T_y)) = \{0\}$. It follows from Proposition 1 that $\sigma(q(T_x)) = \{0\}$ for all $x \in (p,s)$. Thus, $\sigma(T_x) \subseteq \{\lambda_1, \dots, \lambda_n\}$ for all such $x$. Fix any $x \in (p,s)$, $x \neq y$. We may assume $x < y$. Now $T \in \mathscr{B}_{x,y}$, and both $\sigma(T_x)$ and $\sigma(T_y)$ are finite. Then by [2, Cor. 5.2], $\sigma(T_x) = \sigma(T_y)$.

Now we prove the main result.

**Theorem 3.** *Assume $T \in \mathscr{B}_{p,s}$. Then $r(T_x)$ is a continuous function on $(p,s)$.*

*Proof.* If $r(T_y) = 0$ for some $y \in (p,s)$, then $r(T_x) = 0$ for all $x \in (p,s)$ by Proposition 1. Thus, we may assume $r(T_x) > 0$ for all $x \in (p,s)$. For each $n \geq 1$, define

$$\varphi_n(w) = n^{-1} \log(\|(T^n)_{w^{-1}}\|).$$

By the Riesz Convexity Theorem [6, Theorem 11, p. 525] $\varphi_n$ is a convex function on $(s^{-1}, p^{-1})$. Now $\varphi_n(w)$ converges pointwise to $\varphi(w) = \log(r(T_{w^{-1}}))$. By hypothesis $\varphi(w)$ has only finite values, and $\varphi$ must be convex on $(s^{-1}, p^{-1})$. It follows from [10, Theorem 3.2] that $\varphi(w)$ is continuous on $(s^{-1}, p^{-1})$, and thus, $r(T_x)$ is continuous on $(p,s)$.

Concerning this result, note that Example I shows that $x \to r(T_x)$ need not be continuous on the closed interval $[p,s]$.

Next we derive some consequences of the continuity of the spectral radius function. The first is a type of upper semicontinuity property for the function $x \to \sigma(T_x)$. For $K$ a compact subset of $\mathbf{C}$, let $\hat{K}$ denote the polynomial convex hull of $K$. E. Stout's book [11] is a good source for information on polynomial convexity.

**Theorem 4.** *Assume $T \in \mathscr{B} = \mathscr{B}_{p,s}$ and fix $x \in (p,s)$. Let $U$ be an open set in $\mathbf{C}$ with $\sigma(T_x)^\wedge \subseteq U$. Then $\exists \delta > 0$ such that $\sigma(T_y) \subseteq U$ whenever $|y - x| < \delta$.*

*Proof.* Suppose no such $\delta$ exists. Then we can choose a sequence $\{y_n\} \subseteq (p,s)$ with $y_n \to x$ such that for each $n$ there exists $\lambda_n \in \sigma(T_{y_n})$ with $\lambda_n \notin U$. The sequence $\{\lambda_n\}$ is contained in the compact set $\sigma_{\mathscr{B}}(T)$, and so some subsequence converges to a number $\lambda \notin U$. We may assume that $\lambda_n \to \lambda$.

Set $K = \sigma(T_x)^\wedge$, and if $q$ is a polynomial, then let $\|q\|_K = \sup\{|q(\mu)| : \mu \in K\}$. Now $\lambda \notin K$, so by definition there is a polynomial $q$ such that $|q(\lambda)| > \|q\|_K = 1$. Choose a positive integer $n$ sufficiently large that $|q^n(\lambda)| > 2$. Set $p = q^n$. Thus,

$$|p(\lambda)| > 2 \quad \text{and} \quad \|p\|_K = 1.$$

Since $r(p(T_x)) = 1$, by the continuity of the spectral radius (Theorem 3) $r(p(T_{y_k})) \to 1$. Also, $p(\lambda_k) \in \sigma(p(T_{y_k}))$ and $p(\lambda_k) \to p(\lambda)$. This provides a contradiction since for $k$ sufficiently large

$$r(p(T_{y_k})) < 2 \quad \text{and} \quad |p(\lambda_k)| > 2.$$

Next we prove a stronger upper semicontinuity property of the map $x \to \sigma(T_x)$ which holds under the assumption that the spectrum of $T$ in the algebra $\mathscr{B}_{p,s}$ is "thin". First we need a preliminary result that uses the continuity of the spectral radius function.

**Lemma 5.** *Assume $T \in \mathscr{B} = \mathscr{B}_{p,s}$ and $\lambda \notin \sigma_{\mathscr{B}}(T)$. Then $d(x) = \operatorname{dist}(\lambda; \sigma(T_x))$ is a continuous function on $(p, s)$.*

*Proof.* For all $x \in (p, s)((\lambda - T)^{-1})_x = (\lambda - T_x)^{-1}$. Thus since

$$\sigma((\lambda - T_x)^{-1}) = \{(\lambda - \mu)^{-1} : \mu \in \sigma(T_x)\},$$

it follows that for $x \in (p, s)$

$$r(((\lambda - T)^{-1})_x) = \sup\{|\lambda - \mu|^{-1} : \mu \in \sigma(T_x)\} = (d(x))^{-1}.$$

By Theorem 3 this function is continuous on $(p, s)$ which proves the lemma.

**Theorem 6.** *Assume $T \in \mathscr{B} = \mathscr{B}_{p,s}$ and that $\sigma_{\mathscr{B}}(T)$ has empty interior. If $x \in (p, s)$ and $U$ is an open set with $\sigma(T_x) \subseteq U$, then $\exists \delta > 0$ such that $\sigma(T_y) \subseteq U$ whenever $|x - y| < \delta$.*

*Proof.* Suppose no such $\delta$ exists. Then there is a sequence $\{y_n\} \subseteq (p, s)$ such that $y_n \to x$ and for each $n$ there exists $\lambda_n \in \sigma(T_{y_n})$ with $\lambda_n \notin U$. We may assume (by taking a subsequence if necessary) that $\{\lambda_n\}$ converges to a number $\mu \notin U$. Let $\varepsilon = \operatorname{dist}(\mu; \sigma(T_x)) > 0$. Set $D = \{\alpha \in \mathbf{C} : |\alpha - \mu| < \varepsilon/4\}$. Since $\sigma_{\mathscr{B}}(T)$ has no interior, we can choose $\lambda \in D$ with $\lambda \notin \sigma_{\mathscr{B}}(T)$. By Lemma 5 $d(y) = \operatorname{dist}(\lambda; \sigma(T_y))$ is a continuous function on $(p, s)$. By the choices of $\varepsilon$ and $\lambda$, $d(x) > \varepsilon/2$. But since $\lambda_n \to \mu$, we have $d(y_n) < \varepsilon/2$ for all $n$ sufficiently large, a contradiction.

Next we turn to some results concerning the situation where $\sigma(T_x)^\frown$ is disconnected or has an isolated point at some $x \in (p, s)$. What we prove is that this property extends to all $y$ in some neighborhood of $x$.

We need the following technical lemma.

**Lemma 7.** *Let $K$ and $J$ be compact subsets of $\mathbf{C}$ with $\hat{K}$ and $\hat{J}$ disjoint. Then there exists $U$ and $V$, open sets with compact closure, such that*

(i) *$\hat{K} \subseteq U$ and $\hat{J} \subseteq V$; and*
(ii) *$\overline{U}^\frown$ and $\overline{V}^\frown$ are disjoint.*

*Proof.* For each $\mu \in \hat{J}$, choose a polynomial $q_\mu$ such that $\|q_\mu\|_K = 1$ and $|q_\mu(\mu)| > 3$. Let

$$V_\mu = \{\lambda \in \mathbf{C} : |q_\mu(\lambda) > 3\}, \quad \text{and}$$
$$U_\mu = \{\lambda \in \mathbf{C} : |q_\mu(\lambda)| < 2\}.$$

Let $\{V_{\mu_1}, \dots, V_{\mu_n}\}$ be a finite cover for $\hat{J}$. For convenience, in the notation that follows the subscript $\mu_k$ will be replaced by $k$. Note that $\hat{J} \subseteq \bigcup_{k=1}^n V_k$ and $\hat{K} \subseteq \bigcap_{k=1}^n U_k$. Choose $U$ with compact closure such that

$$\hat{K} \subseteq U \subseteq \overline{U} \subseteq \bigcap_{k=1}^n U_k .$$

If $\lambda \in \overline{U}^\frown$, then $\|q_k\|_{\overline{U}} < 2$, so $|q_k(\lambda)| < 2$ for $1 \le k \le n$. Thus, $\overline{U}^\frown \subseteq \bigcap_{k=1}^n U_k$. Since $\bigcap_{k=1}^n U_k$ and $\bigcup_{k=1}^n V_k$ are disjoint, it follows that $\overline{U}^\frown$ and $\hat{J}$ are disjoint.

Now repeat the argument above with $\overline{U}$ in place of $J$ and $J$ in place of $K$. The argument proves $\exists V$ an open set with compact closure such that $\hat{J} \subseteq V$ and $\overline{V}^\frown$ and $\overline{U}^\frown$ are disjoint.

**Theorem 8.** *Let $T \in \mathscr{B}_{p,s}$. Assume for some $x \in (p,s)$, $\sigma(T_x)^\frown$ is disconnected. Then $\sigma(T_y)$ is disconnected for all $y$ in some neighborhood of $x$.*

*Proof.* Assume $K$ and $J$ are disjoint nonempty compact sets with $\sigma(T_x)^\frown = K \cup J$. Then $\hat{K} = K$ and $\hat{J} = J$, so by Lemma 7, there exist open sets $U$ and $V$ with compact closure such that $K \subseteq U$, $J \subseteq V$, and $\overline{U}^\frown$ and $\overline{V}^\frown$ are disjoint. By Theorem 4 $\exists \delta > 0$ such that $\sigma(T_y) \subseteq U \cup V$ whenever $|x-y| < \delta$. Fix $y$ with this property, and assume (without loss of generality) that $x < y$. Then $T \in \mathscr{B} = \mathscr{B}_{x,y}$. By [2, Theorem 5.1(4)]

$$\sigma_{\mathscr{B}}(T) \subseteq \sigma_{\mathscr{B}}(T)^\frown = (\sigma(T_x) \bigcup \sigma(T_y))^\frown \subseteq \left(\overline{U} \bigcup \overline{V}\right)^\frown = \overline{U} \bigcup \overline{V},$$

where the last equality follows from [11, Lemma 29.21(b)]. Now since $\sigma(T_x)$ has nonempty intersection with both $U$ and $V$, the same is true for $\sigma_{\mathscr{B}}(T)$. Then using the fact that $S \to S_y$ is a continuous algebra monomorphism of $\mathscr{B}$ into $\mathscr{B}(L^y)$, it follows from [2, Theorem 4.5] that $\sigma(T_y) \cap U$ and $\sigma(T_y) \cap V$ are nonempty. Thus, $\sigma(T_y)$ is disconnected.

**Corollary 9.** *Let $T \in \mathscr{B}_{p,s}$. Assume for some $x \in (p,s)$ that $\lambda_0$ is an isolated point of $\sigma(T_x)^\frown$. Then $\lambda_0$ is an isolated point of $\sigma(T_y)$ for all $y$ in some neighborhood of $x$.*

*Proof.* Set $K = \{\lambda_0\}$ and $J = \sigma(T_x)^\frown \backslash \{\lambda_0\}$. Choose $U, V, \delta > 0, y$, and $\mathscr{B}$ is in the proof of Theorem 8. Then $\sigma_{\mathscr{B}}(T) \subseteq \overline{U} \cup \overline{V}$. Let $Q \in \mathscr{B}$ be the spectral projection corresponding to the set $\sigma_{\mathscr{B}}(T) \cap \overline{U}$ which is a nonempty open and closed subset of $\sigma_{\mathscr{B}}(T)$. Then by the Spectral Mapping Theorem

$$\sigma((QT)_x) = \sigma(Q_x T_x) = \{\lambda_0\}.$$

It follows from Corollary 2 that for all $w \in (x,y)$, $\sigma((QT)_w) = \{\lambda_0\}$. Therefore $\lambda_0$ is an isolated point of $\sigma(T_w)$ when $x < w < y$. This argument proves that $\lambda_0$ is an isolated point of $\sigma(T_w)$ whenever $|x - w| < \delta$.

*Added in proof.* It has been pointed out to me that a general upper semicontinuity property of the map $y \to \sigma(T_y)$ has been proved by I. Ya. Sneiberg in

[Mat. Issled., 9 (1974), 214–229 (in Russian)]. Both Theorems 4 and 6 are special cases of this result. On the positive side, this same result implies that Theorem 8 and Corollary 9 have more general forms. Specifically, $\sigma(T_x)^\wedge$ can be replaced by $\sigma(T_x)$ in the hypotheses of these two results without altering the conclusions.

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