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QUOTIENTS OF BOUNDED OPERATORS

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Dedicated to Professor Masahiro Nakamura on his 70th birthday

ABSTRACT. We define a quotient $[B/A]$ of bounded operators A and B on a Hilbert space H with $\ker A \subset \ker B$ as the mapping $Ax \to Bx$, $x \in H$, and show explicit formulae for computing quotients which correspond to sums, products, adjoints and closures of given quotients.

1. INTRODUCTION

Let A and B be bounded (linear) operators on a Hilbert space H with the kernel condition

$$\ker A \subset \ker B. \tag{1.1}$$

We then define the quotient $[B/A]$ as the mapping $Ax \to Bx$, $x \in H$. If we write $G(A,B)$ for the set $\{(Ax,Bx); x \in H\}$ in the product Hilbert space $H \times H$, then $G(A,B)$ is a graph. We could define $[B/A]$ as the operator corresponding to this graph. A quotient (of bounded operators) so defined is, as a matter of fact, just what was called "opérateur J uniforme" by Dixmier [2] and "semiclosed operator" by Kaufman [7], and several characterizations were given. It was proved in [7] that the class of quotients contains all closed operators in H and is itself closed under sums and products.

In this paper we first show explicit formulae for computing quotients corresponding to the sum and the product of two given quotients, which really assures the algebraic closure property. We next present reasonable quotients which coincide with the adjoint and the closure of a given quotient if they exist, and deduce a fractional representation due to Kaufman [6] of a densely

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defined closed operator. We also give some other applications which show the advantage of fractional representations of operators.

2. Preliminaries

First we state the majorization lemma due to Douglas [3] which is a key fact for our discussion.

Lemma 2.1 [4, Theorem 2.1]. *Let R and S be bounded operators (on H). Then the following conditions are equivalent.*

(1) $RH \subset SH$.
(2) $RR^* \leq \alpha SS^*$ *for some* $\alpha > 0$.
(3) *There exists a bounded operator X such that $SX = R$.*

With the restriction $\ker X^* \supset \ker S$, the solution X of $SX = R$ in the above lemma is unique [4, p. 259]. We shall call this operator X the Douglas solution, or shortly, D-solution.

It is useful to define the relation $[B/A] \subset [D/C]$ between two quotients $[B/A]$ and $[D/C]$, if $G(A, B) \subset G(C, D)$. In this case we say that $[D/C]$ is an extension of $[B/A]$, or $[B/A]$ is a restriction of $[D/C]$. Note that $G(A, B)$ and $G(C, D)$ are respectively the ranges of operators $\left(\begin{smallmatrix} A & 0 \\ B & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} C & 0 \\ D & 0 \end{smallmatrix}\right)$ on the Hilbert space $H' := H \times H$. Hence, applying Lemma 2.1 to those operators, we can obtain

Lemma 2.2. $[B/A] \subset [D/C]$ *if and only if there exists a bounded operator X such that $A = CX$ and $B = DX$.*

To construct quotients corresponding to the sum and the product of two quotients, we need to show that for bounded operators A and B the sets of the form

$$AH + BH, \quad AH \bigcap BH \quad \text{and} \quad B^{-1}(AH) := \{x \in H ; Bx \in AH\}$$

are operator ranges, that is, ranges of some bounded operators. To this end we shall make use of a technique introduced by Fillmore and Williams [4]. Let R (or more precisely $R_{A,B}$) $= (AA^* + BB^*)^{1/2}$. Then as is well-known $RH = AH + BH$ [4, Theorem 2.2]. To obtain the remaining two kinds of operators we have to use the parallel sum of AA^* and BB^*. Let X and Y be the D-solutions of $RX = A$ and $RY = B$, respectively. Following [4] we put $AA^* : BB^* = AX^*YB^*$ and call it the parallel sum of AA^* and BB^*. Denote by P the orthogonal projection onto the closure $(RH)^-$ of RH. Then (3) and (4) in the following lemma present the desired operators.

Lemma 2.3 (cf. [4, Theorem 4.2]). *Let A and B be bounded operators, and let R, X, Y and P be the operators as given above. Then*

(1) $XX^* + YY^* = P$.
(2) $AA^* : BB^* = BB^* : AA^* = A(1 - X^*X)A^* = B(1 - Y^*Y)B^*$.
(3) $(AA^* : BB^*)^{1/2}H = AH \cap BH$.
(4) $(1 - Y^*Y)^{1/2}H = B^{-1}(AH \cap BH) = B^{-1}(AH)$.

Proof. (1) Put $Q = XX^* + YY^*$. Then by definition $RQR = AA^* + BB^* = R^2$ or $R(Q-P)R = 0$, that is, $(Q-P)RH \subset \ker R$. Since $\ker X^* \supset \ker R$, we have $XH \subset (\ker R)^\perp$, the orthogonal complement of $\ker R$. Similarly we have $YH \subset (\ker R)^\perp$. Hence $(Q-P)RH \subset (Q-P)H \subset (\ker R)^\perp$, which implies that $(Q-P)R = 0$ or $R(Q-P) = 0$. Again by a similar argument as above we can obtain $Q - P = 0$.
(2) Note that $AA^* : BB^* = RXX^*YY^*R$. Since $PX = X$ and $PY = Y$, we can see from (1) that XX^* and YY^* commute. Now all equalities are easily obtained.
(3) Since $SH = (SS^*)^{1/2}H$ in general, it follows from (2) that

$$(AA^* : BB^*)^{1/2}H = A(1 - X^*X)^{1/2}H = B(1 - Y^*Y)^{1/2}H. \tag{2.1}$$

Hence we easily have $(AA^* : BB^*)^{1/2}H \subset AH \cap BH$. To see the converse inclusion, let $y \in AH \cap BH$. Then there exist $u, v \in H$ such that $y = Au = Bv$. Hence $R(Xu - Yv) = 0$, so that $X^*(Xu - Yv) = 0$. Hence

$$\begin{aligned} u &= (1 - X^*X)u + X^*Yv \in (1 - X^*X)H + X^*YH \\ &= \{(1 - X^*X)^2 + (X^*Y)(X^*Y)^*\}^{1/2}H = (1 - X^*X)^{1/2}H. \end{aligned}$$

This implies that $y \in A(1 - X^*X)^{1/2}H$.
(4) We can easily see that $B^{-1}(AH \cap BH) = B^{-1}(AH)$. For another equality, we first have, from (2.1) and (2), that

$$(1 - Y^*Y)^{1/2}H \subset B^{-1}((AA^* : BB^*)^{1/2}H) = B^{-1}\left(AH \bigcap BH\right).$$

Next, if $u \in B^{-1}(AH \cap BH)$ then $Bu \in B(1 - Y^*Y)^{1/2}H$, or $u = (1 - Y^*Y)^{1/2}v + w$ for some $v \in H$ and $w \in \ker B$. Hence we now have only to show that $w \in (1 - Y^*Y)^{1/2}H$. But we can see this from the fact that $(B^*H)^- = (Y^*H)^-$ or $\ker B = \ker Y$.

3. Sums and products

Let $[B/A]$ and $[D/C]$ be two quotients (with the respective kernel conditions of (1.1)). Then the sum of them is defined on the intersection of their domains AH and CH. Hence we naturally choose $A * C := (AA^* : CC^*)^{1/2}$ as the denominator of the sum. Now, for the quotient form of the sum we have

Theorem 3.1 (cf. [7, Theorem 3]). $[B/A] + [D/C] = [(BC_1 + DA_1)/A * C]$.

Here C_1 and A_1 are, respectively, the D-solutions X and Y of $AX = A*C$ and $CY = A * C$.

Proof. Note that $[BX/A * C] = [BX/AX]$ and $[DY/A * C] = [DY/CY]$ are, respectively, restrictions of $[B/A]$ and $[D/C]$ to their common domain $(A * C)H$. Hence the sum of the two quotients is the mapping $(A * C)u \to BXu + DYu$, $u \in H$.

Remark. In the above theorem, clearly we can adopt any operator E with $EH = AH \cap CH$ instead of $A * C$. To construct the corresponding numerator of the sum, we can choose arbitrary solutions X and Y, respectively, instead of the D-solutions, because BX and DY are well-defined by the respective kernel conditions for the two quotients.

The product $[B/A][D/C]$ is defined as the operator $Cu \to Bv$ for $u, v \in H$ such that $Du = Av$. Hence the domain of the product is $\{Cu; u \in D^{-1}(AH)\} = CD^{-1}(AH)$. Let $R_{A,D} = (AA^* + DD^*)^{1/2}$ and let Y be the D-solution of the equation $R_{A,D}Y = D$. Then from Lemma 2.3(4) (replacing B by D), we have $D^{-1}(AH) = (1 - Y^*Y)^{1/2}H$, so that $CD^{-1}(AH) = CA_2H$ if we put $A_2 = (1 - Y^*Y)^{1/2}$. Now, for the product of two quotients we have

Theorem 3.2 (cf. [7, Theorem 3]). $[B/A][D/C] = [BD_2/CA_2]$.

Here A_2 is the operator as given above and D_2 is the D-solution Z of $AZ = DA_2$.

Proof. Since the domain of the product $[B/A][D/C]$ is CA_2H, we see that the composition

$$CA_2u \to DA_2u = AZu \to BZu, \qquad u \in H,$$

determines the desired product.

Remark. In Theorem 3.2 we can replace A_2 by any operator F satisfying $FH = D^{-1}(AH)$. We can also choose any solution Z instead of the D-solution for the same reason as in the case of the sum.

4. Adjoints and closures

Let $[B/A]$ be a quotient densely defined, that is, a quotient with the domain AH dense in H, and let $G(A, B)$ be its graph. Then the adjoint $G(A, B)^*$ is naturally defined as the set of elements (x, y) in $H \times H$ such that $\langle Bu, x\rangle = \langle Au, y\rangle$ for all $u \in H$. ($\langle \cdot, \cdot \rangle$ is the inner product in H.) We can see that $G(A, B)^* = \{(x, y); B^*x = A^*y\}$ and it is a graph again. The corresponding operator is just the adjoint of $[B/A]$, that is, $[B/A]^*$. The domain of $[B/A]^*$ is hence $B^{*(-1)}(A^*H)$, so that by Lemma 2.3(4) the set is the range of the operator $A_* := (1 - Y^*Y)^{1/2}$, where Y is the D-solution of the equation $R_{A^*,B^*}Y = B^*$ ($R_{A^*,B^*} = (A^*A + B^*B)^{1/2}$). For the convenience sake we write $R_l = R_{A^*,B^*}$ and denote by B_l the (unique) solution Z of $ZR_l = B$, $\ker Z \supset \ker R_l$. Then it is clear that $Z = Y^* = B_l$, and hence $A_* = (1 - ZZ^*)^{1/2} = (1 - B_lB_l^*)^{1/2}$. (Similarly to B_l we denote by A_l the solution W of $WR_l = A$, $\ker W \supset \ker R_l$.) Note that $\ker A^* = \{0\}$. Hence we can obtain a bounded operator X uniquely determined by $B^*A_* = A^*X$, and then the mapping $x = A_*u \to Xu = y$, $u \in H$ is just the adjoint $[B/A]^*$. Unlike A_* we define B_* as the operator X (not $(1-A_lA_l^*)^{1/2}$) for formal construction of the quotient representing $[B/A]^*$. Now we have

Theorem 4.1. *Let $[B/A]$ be a densely defined quotient. Then $[B/A]^* = [B_*/A_*]$. Here $A_* = (1-B_lB_l^*)^{1/2}$ and B_* is the unique solution X of $A^*X = B^*A_*$. Explicitly, $B_* = V_lB_l^*$, where V_l is the partial isometry obtained from the polar decomposition $A_l = V_l(A_l^*A_l)^{1/2}$ of A_l.*

Proof. It suffices to show that $B_* = V_lB_l^*$. Denote by P_l the orthogonal projection onto $(R_lH)^-$. Then by Lemma 2.3(1) we can see

$$A_l^*A_l + B_l^*B_l = P_l. \tag{4.1}$$

By definition $A_lR_l = A$ and $B_lR_l = B$. Furthermore, we easily have $(A_l^*A_l)^{1/2} = A_l^*V_l$. Hence now

$$\begin{aligned} A^*B_* = B^*A_* = R_lB_l^*(1-B_lB_l^*)^{1/2} &= R_l(P_l - B_l^*B_l)^{1/2}B_l^* \\ = R_l(A_l^*A_l)^{1/2}B_l^* &= R_lA_l^*V_lB_l^* = A^*V_lB_l^*. \end{aligned}$$

Hence $B_* = V_lB_l^*$, because $\ker A^* = \{0\}$.

Remark. Let C and D be bounded operators such that $CH = A_*H$ and $B^*C = A^*D$. Then by the definition of the adjoint we can see $[B/A]^* = [D/C]$.

To construct the second adjoint $[B/A]^{**} = [B_*/A_*]^*$ we have to assume that A_*H is dense in H. Since $[B/A]^{**}$ is then the closure of $[B/A]$, this assumption is nothing but the condition for $[B/A]$ to be closable. First, related to this condition we have

Theorem 4.2. *Let $[B/A]$ be a densely defined quotient. Then the following conditions are equivalent.*

(1) *$[B/A]$ is closable. (A_*H is dense in H.)*
(2) $\ker A_l \subset \ker B_l$.
(3) $A_*^2 + B_*^*B_* = 1$.

Proof. (1) $\Rightarrow$ (2). Let $A_lu = 0$. Then from (4.1) we have $A_*^2B_lu = (1-B_lB_l^*)B_lu = B_l(P_l - B_l^*B_l)u = B_lA_l^*A_lu = 0$. Hence $B_lu \in \ker A_* = \{0\}$.
(2) $\Rightarrow$ (3). Since $\ker A_l = (1-V_l^*V_l)H$, we have $B_l(1-V_l^*V_l)H = \{0\}$, or $B_l = B_lV_l^*V_l$. Hence $A_*^2 + B_*^*B_* = 1 - B_lB_l^* + B_lV_l^*V_lB_l^* = 1$.
(3) $\Rightarrow$ (1). Let $A_*u = 0$. Then we have to show $u = 0$. Since $\ker A_* \subset \ker B_*$, we have $B_*u = 0$, so that $u = (A_*^2 + B_*^*B_*)u = 0$.

Remark. Every quotient is not closable; so let P_* be the orthogonal projection onto $(A_*H)^-$. Then with the same assumption as in Theorem 4.2 we have

$$A_*^2 + B_*^*B_* = P_*. \tag{4.2}$$

Proof. From the identity $B_* = V_lB_l^*$ and (4.1) we can see $B_*A_* = A_lB_l^*$. Using this fact and (4.1) again, we can show that $A_*(A_*^2 + B_*^*B_*)A_* = A_*^2$, or

$A_*QA_* = 0$, where $Q = A_*^2 + B_*^*B_* - P_*$. On the other hand we easily see that $QH \subset (\ker A_*)^\perp$. Hence we can obtain $Q = 0$, say, by a similar argument as in the proof of Lemma 2.3(1).

Now, on the second adjoint or closure of a quotient we have

Theorem 4.3. *Let $[B/A]$ be a densely defined, closable quotient. Then $[B/A]^{**} = [B_*^*/(1 - B_*B_*^*)^{1/2}] = [B_l/A_l]$.*

Proof. By definition $[B/A]^{**} = [B_*/A_*]^* = [B_{**}/A_{**}]$. By Theorem 4.2(3) we can see that $A_{*l}(= (A_*)_l) = A_*$ and $B_{*l} = B_*$, so that $A_{**} = (1 - B_*B_*^*)^{1/2}$. Further, let X be the unique solution of $A_*X = B_*^*A_{**}$. Then, since $A_*X = B_*^*(1 - B_*B_*^*)^{1/2} = (1 - B_*^*B_*)^{1/2}B_*^* = A_*B_*^*$, we have $X = B_*^*$, or $B_{**} = B_*^*$ by definition. Hence we have the first equality. For the other equality we first note that $[B_l/A_l]$ is a quotient by Theorem 4.2(2). Next since A_lH is dense in H, we have $V_lV_l^* = 1$. From this identity and (4.1) we can see that $A_lH = (V_lA_l^*A_lV_l^*)^{1/2}H = (1 - B_*B_*^*)^{1/2}H$ and $A_*B_l = B_*^*A_l$. Hence by the remark after Theorem 4.1 we have $[B_*/A_*]^* = [B_l/A_l]$.

Remarks. (1) If $[B/A]$ is closable, then $G(A_l, B_l)$ is a graph by Theorem 4.2(2). Hence we could show $[B/A]^{**} = [B_l/A_l]$ from the fact $G(A_l, B_l) = G(A, B)^-$.
(2) If $[B/A]$ is closed, then R_lH is closed in H [6, Theorem 1]. In this case we have $A_l = AX$ and $B_l = BX$ for a generalised inverse X of R_l (say, [1, p. 321]). Hence we can easily see that $\ker A_l \subset \ker B_l$ and $[B/A] = [B_l/A_l]$ (say, by Lemma 2.2).

When an operator T is densely defined and closed, then T is represented as a quotient (say, [11, p. 307]). Since $T = T^{**}$, we can write $T = [B_*^*/(1 - B_*B_*^*)^{1/2}]$ with some operator B_* by Theorem 4.3. Hence, putting $B_*^* = C$, we now obtain the following fractional representation of a closed operator due to Kaufman. (His notation is $C(1 - C^*C)^{-1/2}$.)

Corollary 4.4 [6, Corollary]. *Let T be a densely defined closed operator. Then there exists a pure contraction C, that is, contraction with $\ker(1 - C^*C) = \{0\}$, such that $T = [C/(1 - C^*C)^{1/2}]$. Furthermore, the operator C is uniquely determined by T.*

Proof. It suffices to show uniqueness of C. Let $[C/(1 - C^*C)^{1/2}] = [D/(1 - D^*D)^{1/2}]$ for two pure contractions C and D. Then $(1 - C^*C)^{1/2} = (1 - D^*D)^{1/2}X$ and $C = DX$ for some invertible operator X by Lemma 2.2. We can easily obtain $X^*X = 1$. Hence X is unitary, so that $1 - C^*C = (1 - D^*D)^{1/2}XX^*(1 - D^*D)^{1/2} = 1 - D^*D$, or $C^*C = D^*D$. Hence $X = 1$, and $C = D$.

With the same notation as before we can show the following fact, say, from Theorem 4.1 and the identity $B_*^*B_* = B_lB_l^*$.

Corollary 4.5 [8, Lemma 1]. $T^* = [C^*/(1 - CC^*)^{1/2}]$.

5. Some applications

In this section we apply quotients to obtain some new results and new proofs of known results.

I. *Bounded quotients.* A quotient $[B/A]$ is a restriction to AH of a bounded operator when $B^*H \subset A^*H$. In fact, if $B^* = A^*X$ or $B = X^*A$ for a bounded operator X, then $[B/A] = [X^*A/A] \subset [X^*/1] = X^*$. We call such a quotient as $[B/A]$ bounded. Motivated by Okazaki [9] and Ôta [10], we show two theorems on bounded quotients.

Theorem 5.1 (cf. [9, Theorem 3]). *Let $[B/A]$ be a quotient with $BH \subset AH$. If there exists $\lambda > 0$ such that $(\mu A - B)H$ is closed in H for $\mu > \lambda$, then $[B/A]$ is bounded.*

Proof. There exists a bounded operator C such that $AC = B$. Hence, if $\mu > \|C\| + \lambda$, then $\mu - C$ is invertible, and $AH = A(\mu - C)\cdot(\mu - C)^{-1}H = (\mu A - B)H$. Hence AH is closed, so that A^*H is closed. Hence $B^*H \subset (\ker B)^\perp \subset (\ker A)^\perp = A^*H$, which implies that $[B/A]$ is bounded.

Theorem 5.2 (cf. [10, Theorem 3.3]). *Let $[B/A]$ be a densely defined closed quotient, and let $[B/A]^* = [B_*/A_*]$ be its adjoint. If $B_*H \subset AH$, then $[B/A]$ is bounded, so that it extends to a bounded operator on H.*

Proof. Since $[B/A]$ is closed, we see that $[B/A] = [B_*^*/(1 - B_*B_*^*)^{1/2}]$ by Theorem 4.3. Hence, by the assumption we have $B_*H \subset AH = (1 - B_*B_*^*)^{1/2}H$, so that $B_*B_*^* \le \alpha(1 - B_*B_*^*)$ for some $\alpha > 0$ by Lemma 2.1. Hence $B_*B_*^* \le \alpha/(1+\alpha) < 1$, and $1 - B_*B_*^*$ is invertible. Hence $[B/A]$ is bounded.

II. *Normal operators.* As one of the preservative properties of the function $\Gamma(C) := C(1 - C^*C)^{-1/2}$ (C is a pure contraction), Kaufman [8] proved that $\Gamma(C)$ is normal if and only if C is normal. Using the product formula of quotients, we give an alternative proof to this fact (and a result in Stone [12]).

Theorem 5.3 (cf. [8, Theorem 2], [12, Theorem 6]). *Let $T = [C/(1 - C^*C)^{1/2}]$ be a densely defined closed operator. Then the following conditions are equivalent.*

(1) *T is normal, that is, $T^*T = TT^*$.*
(2) $T^*T \subset TT^*$.
(3) $T^*T \supset TT^*$.
(4) *C is normal.*

Proof. By the formula of the product we can see that $T^*T = [C^*/(1-CC^*)^{1/2}]\times[C/(1 - C^*C)^{1/2}] = [C^*C/(1 - C^*C)]$. Similarly, we can also see that $TT^* = [CC^*/(1-CC^*)]$. Hence, if $T^*T \subseteqq TT^*$, then we have $1-C^*C = (1-CC^*)X$ and $C^*C = CC^*X$ for some operator X. We now immediately have $X = 1$. Hence C is normal. Conversely, if C is normal, then clearly $T^*T = TT^*$. In case $T^*T \supset TT^*$, we can still obtain the same conclusion as before.

III. *Decomposition into the closable part and the singular part.* In [5] Jorgensen introduced the decomposition of a (densely defined) operator T into the sum of its closable part $T_c := QT$ and its singular part $T_s := Q^{\perp}T$ $(Q^{\perp} = 1 - Q)$, where Q is the orthogonal projection onto the closure of the domain of T^* [5, p. 285]. If $T = [B/A]$, a densely defined quotient, then we have $Q = P_* = [P_*/1]$, $T_c = [P_*/1][B/A] = [P_*B/A]$ and $T_s = [P_*^{\perp}/1][B/A] = [P_*^{\perp}B/A]$. Furthermore, using (4.1) and (4.2) we can obtain that $P_*B = B_lV_l^*V_lR_l$. Hence we have

Theorem 5.4. *Let* $[B/A]$ *be a densely defined quotient. Then*

$$\begin{aligned}[B/A] &= [P_*B/A] + [P_*^{\perp}B/A] \\ &= [B_lV_l^*V_lR_l/A] + [(B - B_lV_l^*V_lR_l)/A]\end{aligned}$$

is its sum decomposition into the closable part and the singular part in the sense of Jorgensen.

Related to the above decomposition we have

$$P_*^{\perp}B = 0 \tag{5.1}$$

as an equivalent condition for $[B/A]$ to be closable. Since $\ker B^* \subset B^{*(-1)}(A^*H) = A_*H \subset P_*H$, we can see that (5.1) (or $B^*P_*^{\perp} = 0$) is really equivalent to the identity $P_*^{\perp} = 0$, or $P_* = 1$, the condition (1) Theorem 4.2.

As an extremely nonclosable quotient we call $[B/A]$ singular if $P_*^{\perp}B = B$. Now we show a characterization of such a quotient.

Theorem 5.5. *Let* $[B/A]$ *be a densely defined quotient. Then the following conditions are equivalent.*

(1) $[B/A]$ *is singular.*
(2) $A_lB_l^* = 0$.
(3) $A^*H \cap B^*H = \{0\}$.

Proof. (1) $\Rightarrow$ (2). Since (1) is equivalent to $BH \subset \ker A_*$, we see that $A_*B = 0$, or $B^*A_* = 0$. Hence $R_lA_l^*A_lB_l^* = R_l(P_l - B_l^*B_l)B_l^* = R_lB_l^*(1 - B_lB_l^*) = B^*A_*^2 = 0$. Hence, using $\ker R_l \subset \ker B_l$, we have $(A_lB_l^*)^*(A_lB_l^*) = B_lA_l^*A_lB_l^* = 0$, so that $A_lB_l^* = 0$.

(2) $\Rightarrow$ (3). From Lemma 2.3(3) we see that (3) is equivalent to

$$A^*A : B^*B = 0. \tag{3'}$$

Hence by the definition $A^*A : B^*B = A^*A_lB_l^*B$, the relation (2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). Note that $(A_*B)^*(A_*B) = B^*A_*^2B = B^*(1 - B_lB_l^*)B = A^*A : B^*B$. Hence, from (3′) we have $A_*B = 0$, that is, $BH \subset \ker A_*$.

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