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FUNCTIONS NOT CONSTANT ON FRACTAL QUASI-ARCS OF CRITICAL POINTS

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ABSTRACT. This paper provides geometric sufficient conditions for an arc to be a critical set for some function not constant along that arc—an example of which was first discovered by Whitney in 1935. In particular, any fractal subarc of a quasi-circle has this property. The maximum degree of differentiability of the function is closely connected to the arc's geometry.

1. INTRODUCTION

If γ is a smooth curve along which a function f is critical (i.e. $\nabla f = 0$ on γ), then integration along γ shows that f must be constant on γ. The same holds true even if γ is only continuous, provided that f is sufficiently differentiable (by an easy application of the Morse-Sard Theorem [10]). However, a celebrated example due to Whitney [13] provides a C^1 function f with a (nonrectifiable) critical arc along which f is *not* constant. The counterintuitive nature of this example may be easily grasped by imagining the graph of such an $f: \mathbf{R}^2 \to \mathbf{R}$ as a C^1 hill containing a path σ that climbs to the top of the hill, but such that at every point of σ the hill has a horizontal tangent plane!

Whitney's example has inspired a great deal of mathematics, beginning with the Whitney Extension Theorem ([12]) and including the work of A. P. Morse and A. Sard on the Morse-Sard Theorem. However, the questions raised by the example have not been fully answered. The principal outstanding question is the following. Call an arc 1-*critical* if there is a C^1 function critical but not constant along it.

Q: *Can the 1-critical arcs be characterized geometrically, and if so, how?*

This paper contains a geometric sufficient (but not necessary) condition for an arc to be 1-critical (Theorem 3). (A different, less geometric sufficient condition appears in [6].)

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A related question was vaguely posed in Whitney's original paper, and can be stated as follows: *Given a function f, how far from rectifiable must a closed connected set be to be a critical set for f on which f is not constant*? This question is answered in [7] by interpreting "distance from rectifiability" by means of the Hausdorff dimension of the set, and relating this to the precise degree of differentiability of f (see Corollary 1 below). The purpose of this paper is to provide some results in the opposite direction, that is, to provide some broad, geometric sufficient conditions which allow the Whitney phenomenon to occur.

A few definitions are necessary to begin the discussion and fix notation. We restrict our attention to the Euclidean spaces $\mathbf{R}^n$, $n \geq 1$, since all our results are essentially local. For each $s > 0$, let μ_s denote the Hausdorff (outer) measure in dimension s (for reference see [4], [9]). We say that a set $B \subset \mathbf{R}^n$ is null if it has Lebesgue measure zero, s-null if $\mu_s(B) = 0$, s-finite if $\mu_s(B) < \infty$, and s-sigmafinite if B is a countable union of s-finite sets. $HD(B)$ will denote the Hausdorff dimension of B, i.e. the unique real number with the property that $\mu_s(B) = 0$ and $\mu_t(B) = \infty$ for all s, t such that $t < HD(B) < s$.

If m, n, k are positive integers, recall that a function $f\colon \mathbf{R}^m \to \mathbf{R}^n$ is of class C^k $(f \in C^k)$ if f and all its partial derivatives of order $\leq k$ are defined and continuous on $\mathbf{R}^m$. This can be extended as follows. For a real number $s \geq 1$, let $[s]$ and $\{s\}$ denote its integer and fractional parts, respectively. We say that f is of class C^s if $f \in C^{[s]}$ and the $[s]$th derivative $D^{[s]}f$ satisfies, for every compact set U,

$$|D^{[s]}f(x) - D^{[s]}f(y)| \leq M|x-y|^{\{s\}}$$

for all $x, y \in U$, where M is a positive constant depending only on U.

Function f is of class $C^{k+\mathrm{Lip}}$ if $D^k f$ satisfies the above condition with $s = k$ and the exponent $\{s\}$ replaced by 1.

Finally, f is of class C^{s+} if $f \in C^s$ and for each compact set U,

$$|D^{[s]}f(x) - D^{[s]}f(y)|/|x-y|^{\{s\}} \to 0$$

as $|x-y| \to 0$, $x, y \in U$.

With these definitions we have interpolated uncountably many distinct "degrees of differentiability" between each positive integer k and $k+1$: for every integer $k \geq 1$ and every s, t such that $k < s < t < k+1$, we have

$$C^{k+1} \subsetneqq C^{k+\mathrm{Lip}} \subsetneqq C^{t+} \subsetneqq C^t \subsetneqq C^{s+} \subsetneqq C^s \subsetneqq C^k.$$

Note also that $C^{s+} = C^s$ in case s is an integer.

2. Theorems

To put the following material into context, we state the following theorem from [7]:

Theorem 1. *Let $f\colon \mathbf{R}^m \to \mathbf{R}^n$ and A be a critical set for f.*

(i) *If $f \in C^{s+}$ and A is $(s+n-1)$-sigmafinite, then $f(A)$ is null.*

(ii) *If* $f \in C^s$ *and* A *is* $(s+n-1)$*-null, then* $f(A)$ *is null.*
(iii) *If* $f \in C^{k+\mathrm{Lip}}$ *and* A *is* $(k+n)$*-null, then* $f(A)$ *is null.*

To introduce some convenient terminology, let A be a connected subset of some Euclidean space, and $s \geq 1$. We shall say that A has *complexity* $\leq s$ if every pair of points in A is contained in a connected s-sigmafinite subset of A. The real number $complexity(A)$ is defined to be the infimum of the set $\{s\colon A$ has complexity $\leq s\}$. Note that any connected subset of $\mathbf{R}^n$ has complexity $\leq n$, while $\mathbf{R}^n$ itself and all smooth connected submanifolds have complexity 1.

Corollary 1. *If* f *is a real-valued function of class* C^{s+} *and* A *is a critical set for* f *with complexity* $\leq s$*, then* f *is constant on* A.

Proof. Let $n = 1$ in Theorem 1(i) and note that $f(A) \subset \mathbf{R}$ is connected and null, hence a single point. □

For A and s as above, we say (after Choquet [3]) that A is *s-critical*, $s \geq 1$, if there exists a real-valued function $f \in C^s$ which is critical but not constant on A. Every set, by convention, is 0-critical. Then we define the real number $criticality(A)$ to be the supremum of the set $\{s\colon A$ is s-critical$\}$. We may now rephrase Corollary 1 as

Corollary 2. *Let* A *be a connected subset of some Euclidean space. Then criticality*$(A) \leq complexity(A)$.

Proof. If not, there are s, t such that $s < t$, A is a t-critical, and A has complexity $\leq s$. Since $C^t \subset C^{s+}$, this contradicts Corollary 1. □

Notice that Corollary 2 relates two quite different notions of the "complexity" of a connected set: one purely geometric, the other purely function-theoretic. One goal of this paper is to establish the reverse inequality and so obtain the equality $complexity(A) = criticality(A)$ (see Corollary 3). However, this is not true in general, even for such well-behaved connected sets as compact arcs. For example, Theorem 1 implies that any rectifiable arc has criticality 0 but complexity 1. Less immediate is a theorem due to Choquet [3] which says that the graph of any continuous function $f\colon \mathbf{R} \to \mathbf{R}$ has criticality 0 (while of course the complexity of any connected set containing more than one point is not less than one).

However, the inequality of Corollary 2 does become equality for a certain subclass of connected sets defined below. A subset γ of Euclidean space is an *arc* if it is the image of a continuous injection defined on the closed unit interval. If γ is an arc and $\phi\colon [0,1] \to \gamma$ is the corresponding injection, for x, $y \in \gamma$ let $\gamma(x,y)$ denote the subarc of γ lying between x and y; i.e. $\gamma(x,y)$ is the image under ϕ of the open interval formed by $\phi^{-1}(x)$ and $\phi^{-1}(y)$ in $[0,1]$. An arc γ is a *quasi-arc* if there is some $K > 0$ such that for every x, $y \in \gamma$, $\gamma(x,y)$ is contained in some ball of radius $K|x-y|$. (This is equivalent to saying that γ is a subarc of an Ahlfors quasi-circle.)

Theorem 2. (a) *Let $s > 1$ be a nonintegral real number. If γ is a quasi-arc with $\mu_s(\gamma) > 0$, then γ is s-critical.*

(b) *If $s > 1$ is an integer, then under the same hypotheses there is a function of class $C^{s-1+\mathrm{Lip}}$ which is critical but not constant on γ.*

(The proofs of this and the next theorem will follows in §3.) Combined with Corollary 2, this yields

Corollary 3. *If γ is a quasi-arc with $HD(\gamma) > 1$ (i.e. if γ is a fractal quasi-arc), then $criticality(\gamma) = complexity(\gamma) = HD(\gamma)$.*

Proof. By Theorem 2, if $1 < s < HD(\gamma)$, then γ is s-critical. Hence *criticality* $(\gamma) \geq HD(\gamma)$. On the other hand, for each $t > HD(\gamma)$, γ is t-null, hence has complexity $\leq t$. Therefore, $complexity(\gamma) \leq HD(\gamma)$. Now apply Corollary 2. □

One result of Theorem 2 is that it shows Corollary 1 to be sharp. But its main interest, aside from Corollary 3, is that it provides all at once a very large class of Whitney-type examples, which up to now had to be explicitly constructed one at a time (see [2], [3], [5], [6]). In reference to question Q from the beginning of this paper, Theorem 2 says that any fractal quasi-arc is 1-critical. (*Fractal* conventionally means a set whose Hausdorff and topological dimensions disagree.) Since fractal quasi-arcs are in plentiful supply (e.g. as Julia sets for certain rational maps in the plane), so are 1-critical sets.

We can refine this sufficient condition somewhat by means of Theorem 3 below. First a definition. An arc γ is a *t-quasi-arc*, $t \geq 1$, if there is a constant $K > 0$ such that for every $x, y \in \gamma, |\gamma(x, y)|^t \leq K|x - y|$. The number t is a *quasi-exponent* for γ. A 1-quasi-arc is just an ordinary quasi-arc; in the more general case t measures the degree to which the arc fails to be a quasi-arc. An arbitrary arc need not be a t-quasi-arc for any $t < \infty$, but it is tempting to think of such examples as unusual cases.

Theorem 3. *Let $s > t \geq 1$. Suppose γ is a t-quasi-arc with positive Hausdorff s-measure. Then $criticality(\gamma) \geq s/t$. In particular, γ is 1-critical.*

Theorem 3 provides a purely geometric sufficient condition for an arc to be 1-critical. Note that, in view of Choquet's result (that a plane arc which is the graph of a continuous function is never 1-critical), this implies that the graph Γ of a continuous function cannot be a t-quasi-arc for any $t < HD(\Gamma)$. (This is not difficult to prove directly.) Hence it would be consistent with Choquet's negative result and Theorem 3 to suppose that some inequality such as $\inf\{t\colon \gamma \text{ is a } t\text{-quasi-arc}\} < HD(\gamma)$ might provide an answer to Q.

At face value, however, this fails to be an adequate necessary condition for an arc to be 1-critical: we may always append a badly non-quasi-arc σ to a fractal quasi-arc γ to form a new arc $\gamma \cup \sigma$. Then the proof of Theorem 2 shows that we may find a C^1 function critical but not constant on $\gamma \cup \sigma$ by virtue of being constant on σ but not on γ. To avoid inessential examples like

these, we might conjecture the following: If f is a C^1 function critical but not constant on γ, then for every subarc $\eta \subset \gamma$ on which f is not constant, $\inf\{t\colon \eta \text{ is a } t\text{-quasi-arc}\} < HD(\gamma)$. Unfortunately even this conjecture fails, as the following proposition (and its proof) shows.

Proposition. *For each $t > 1$ there is a t-quasi-arc γ in $\mathbf{R}^2$ (so that $HD(\gamma) \leq 2$) with the properties:*

(a) *γ is not an s-quasi-arc for any $s < t$, and*
(b) *there exists a function $f\colon \mathbf{R}^2 \to \mathbf{R}$ such that $f \in C^s$ for all $s < (t+1)/t$, γ is a critical set for f, and $f[\gamma] = [0,1]$.*

Idea of proof (for full details see [8]):
For fixed $t > 1$, create an arc in the shape of a comb on which each tooth has smaller teeth, these smaller teeth themselves have teeth, etc. (see Figure 1).

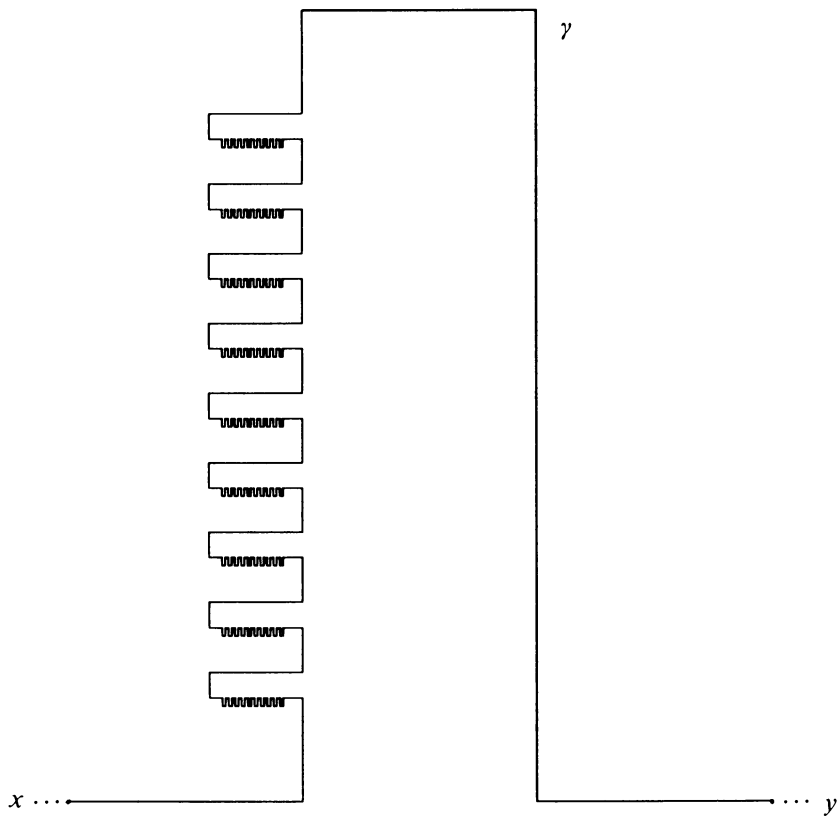

FIGURE 1.
A tooth γ of the "comb," with endpoints x and y, having teeth of its own, each of which itself has teeth, etc.

At each level the diameter of a tooth $|\gamma|$ is related to the distance between its endpoints x, y by $|\gamma|^t = |x - y|$. Assuming the limiting result is a continuous arc, it will clearly be a t-quasi-arc with property (a). If there are sufficiently

many teeth at each level, the arc will have Hausdorff dimension greater than one. A natural Cantor function defined along the arc which is constant on rectifiable subarcs (i.e. subarcs disjoint from the Cantor set of limits of descending sequences of teeth) turns out to be the required function f.

This proposition shows that there are arcs in the plane with arbitrarily high (but finite) quasi-exponents but which are nevertheless 1-critical. Hence any satisfactory answer to Q will have to involve ideas beyond quasi-exponents and Hausdorff dimension.

The following question remains open: Is there an arc γ and a C^1 function f critical but not constant on γ such that, for every subarc η of γ on which f is not constant, η fails to be a t-quasi-arc for any $t \in (1, \infty)$? If not, this would provide another negative result to place next to Choquet's theorem.

3. Proofs

The proof of Theorem 2 requires two essential lemmas, both important theorems in their own right.

Lemma 1. *Let E be a closed subset of $\mathbf{R}^n$ with $\mu_s(E) > 0$. Then there is a compact subset F of E such that $\mu_s(F) > 0$ and, for some constant $M > 0$,*

$$\mu_s(B_r(x) \cap F) \leq M r^s \quad \textit{for all } x \in \mathbf{R}^n \textit{ and all } r \leq 1 ,$$

where $B_r(x)$ denotes the open ball of radius r, center x.

Lemma 1 is a slight variation of a bounded density theorem apparently first stated by Falconer [4, Theorem 5.4(b)], which follows in turn by a small modification of the proof of a fundamental theorem due to Besicovitch [1] (showing that closed sets of infinite s-measure always contain closed subsets of positive, finite s-measure). The interested reader may find all the necessary ideas for the proof clearly laid out in [4], or may consult [8] for the explicit details.

Lemma 2 (Extended Whitney Extension Theorem). *Let A be a closed subset of $\mathbf{R}^n$, B any Banach space, and $f: A \to B$. Let $r \geq 1$ be an integer, and suppose $f_0, f_1, \dots, f_r$ are functions ("candidate derivatives" for f) such that $f_0 = f$ and for each $k = 1, \dots, r$,*

$$f_k : A \to L^k(\mathbf{R}^n, B).$$

($L^k(\mathbf{R}^n, B) =$ the symmetric k-linear maps from $\mathbf{R}^n$ to B.) Define $R_k : A \times A \to L^k(\mathbf{R}^n, B)$ by

$$f_k(y) = \sum_{i=k}^{r} f_i(x)(y-x)^{i-k}/(i-k)! + R_k(x, y) \quad \textit{for all } x, y \in A .$$

(a) *Let s be a (nonintegral) real number such that $r < s < r+1$. Suppose that for each $x \in A$ and each $k = 0, 1, \dots, r$,*

$$|R_k(y, z)| \leq N_k |y - z|^{s-k} \quad \textit{for all } y, z \in B_1(x) \cap A ,$$

where N_k *is some positive constant depending only on* x *and* k. *Then* f *extends to a* C^s *function* $G\colon \mathbf{R}^n \to B$ *such that*

$$D^k G(x) = f_k(x) \quad \textit{for all } x \in A\,, \textit{ and each } k = 0, 1, \dots, r\,.$$

(b) *Suppose* $s = r + 1$ *and the hypotheses of* (a) *hold. Then* f *extends to a* $C^{r+\mathrm{Lip}}$ *function* $G\colon \mathbf{R}^n \to B$ *such that*

$$D^k G(x) = f_k(x) \quad \textit{for all } x \in A \textit{ and each } k = 0, 1, \cdots, r\,.$$

The proof of Lemma 2 is similar to the proof of Whitney's original theorem for C^k maps [12], but requires more work. For part (a) see [11] or [8]; the proof of part (b) is entirely analogous.

We will need only the following simple consequence:

Corollary 4. (a) *Let* A *be a compact subset of* $\mathbf{R}^m$, $s > 1$ *be a nonintegral real number, and* $f\colon A \to \mathbf{R}^n$ *be a function with the property that for some* $M > 0$,

$$\frac{|f(x) - f(y)|}{|x - y|^s} \le M \quad \textit{for all } x, y \in A\,,\ x \ne y\,.$$

Then there is a function $G\colon \mathbf{R}^m \to \mathbf{R}^n$ *of class* C^s *such that* G *and* f *agree on* A, *and* $DG(x) = 0$ *for all* $x \in A$.

(b) *Same conclusion for* $s > 1$ *an integer, with* G *of class* $C^{s-1+\mathrm{Lip}}$.

Proof. Apply Lemma 2 with $f_0 = f$ and $f_k = 0, k = 1, \dots, [s]$ for part (a) (respectively, $k = 1, \dots, s - 1$ for part (b)). Then $R_k = 0$ for $k \ge 1$ and $R_0(x, y) = f(y) - f(x)$, so the conclusion follows immediately. □

It is worth remarking here that a direct proof of Corollary 4 can be given that uses the same ideas as the proof of Lemma 2 but is considerably simpler because most of the computational difficulties related to keeping track of the derivatives are absent. The advantage of a separate proof of this special case is that it brings the main ideas of Whitney's proof into sharper relief and so serves as a good introduction to the full theorem.

For the reader's convenience, here is a sketch of the main steps of a direct proof of Corollary 4:

(i) Partition $\mathbf{R}^m \backslash A$ into a union of squares Q_i such that the distance $d(Q_i, A)$ from Q_i to A is comparable to the diameter of Q_i.

(ii) Construct a partition of unity $\{\phi_i\}$ based on $\{Q_i\}$ satisfying

$$|D^k \phi_i(y)| \le C d(y, A)^{-k} \quad \text{for all } i, k \text{ and some } C > 0\,.$$

(iii) For each i pick $x_i \in A$ so that $d(x_i, Q_i) = d(A, Q_i)$.

(iv) Define $G\colon \mathbf{R}^m \to \mathbf{R}^n$ by $G(y) = f(y)$ for $y \in A$,

$$G(y) = \sum_j \phi_j(y) f(x_j) \quad \text{for } y \notin A\,.$$

(v) Now prove by induction on k that $G \in C^k$ and $D^k G = 0$ on A for $k = 1, \dots, [s]$.

(vi) Show $D^{[s]}G$ is $\{s\}$-Hölder by considering cases and using the triangle inequality and the mean-value theorem.

Proof of Theorem 2. (a) Since γ is closed, apply Lemma 1 to obtain a compact subset F of γ and $M > 0$ such that $0 < \mu_s(F) < \infty$ and $\mu_s(B_r \cap F) \le Mr^s$ for all $r \le 1$ and all balls B_r of radius r.

Let a be an endpoint of γ, and define $f\colon \gamma \to \mathbf{R}$ by

$$f(x) = \mu_s(\gamma(a, x) \cap F) .$$

Then f is not constant since $\mu_s(F) > 0$, and when $|x - y|$ is small,

$$\begin{aligned}\frac{|f(x) - f(y)|}{|x - y|^s} &= \frac{\mu_s(\gamma(x, y) \cap F)}{|x - y|^s} \\ &\le \frac{\mu_s(B_{K|x-y|} \cap F)}{|x - y|^s} \quad \text{for some } K > 0 \text{ and some ball } B_{K|x-y|}, \text{ since } \gamma \text{ is a quasi-arc} \\ &\le \frac{M(K|x - y|)^s}{|x - y|^s} = MK^s.\end{aligned}$$

Hence Corollary 4 yields a C^s function G which extends f and is critical on γ. Since G is not constant on γ, γ is s-critical.

(b) The same argument, using Corollary 4(b), will suffice. □

An easy modification of this argument yields the proof of Theorem 3.

Proof of Theorem 3. Let f, F, K, and M be as above in the proof of Theorem 2. Then

$$\begin{aligned}\frac{|f(x) - f(y)|}{|x - y|^{s/t}} &= \frac{\mu_s(\gamma(x, y) \cap F)}{|x - y|^{s/t}} \\ &\le \frac{\mu_s(B_{(K|x-y|)^{1/t}} \cap F)}{|x - y|^{s/t}} \\ &\le \frac{M(K|x - y|)^{s/t}}{|x - y|^{s/t}} \quad \text{as before} \\ &\le MK^{s/t}.\end{aligned}$$

If s/t is not an integer, Corollary 4(a) immediately yields the desired result. If s/t is an integer, Corollary 4(b) yields a function G of class $C^{(s/t)-1+\mathrm{Lip}}$ critical but not constant on γ. Since

$$G \in C^{(s/t)-1+\mathrm{Lip}} \subset C^\beta \quad \text{for all } \beta < s/t ,$$

γ is β-critical for all $\beta < s/t$, and so

$$s/t \le \sup\{\beta : \gamma \text{ is } \beta\text{-critical}\} = \mathit{criticality}(\gamma) .$$

□

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