MULTI-STATES ON $C^*$-ALGEBRAS

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Abstract. This paper is concerned with the study of the dual of a $C^*$-algebra as a matrix ordered space. It is shown that an $n \times n$ matrix of linear functionals of a $C^*$-algebra, satisfying the generalized positivity condition, induces a representation of the algebra that generalizes the classical Gelfand-Naimark-Segal representation. This allows analysis of the relationship between the comparability of cyclic representations of the algebra and the matricial order structure of the dual. We consider the problem of unitary diagonalization of linear functionals and show that positive normal functionals on a matrix algebra over a semifinite von Neumann algebra can always be diagonalized.

1. Introduction and preliminaries

The dual of a $C^*$-algebra has the natural matrix order. An $n \times n$ matrix $[\rho_{jk}]$ of linear functionals on a $C^*$-algebra $\mathcal{A}$ is said to be an $n$-positive linear functional on $\mathcal{A}$ if $[\rho_{jk}(A_{jk})]$ is a positive $n \times n$ matrix whenever $[A_{jk}]$ is a positive element of the $C^*$-algebra of $n \times n$ matrices over $\mathcal{A}$. If $[\rho_{jk}]$ satisfies the normalization condition $\rho_{jj}(I) = 1$ for each $j$ in $\{1, \ldots, n\}$, we say that $[\rho_{jk}]$ is an $n$-state (multi-state, when $n$ is not specified).

In this paper we develop basic theory of multi-states, and our approach is motivated by the case of a single state. Thus with each $n$-positive linear functional of $\mathcal{A}$ we associate a representation of $\mathcal{A}$ that generalizes the classical GNS representation (Theorem 2.1). This enables to clarify the relationship between the comparability of cyclic representations of $\mathcal{A}$ and the matricial order structure of $\mathcal{A}^*$. We characterize irreducible and factor representations engendered by multi-states in terms of component functionals, and use this characterization to locate a certain class of extreme points in the set of identity preserving completely positive maps of $\mathcal{A}$ into $M_n(C)$. Finally, we consider the problem of unitary diagonalization of linear functionals on a matrix algebra over a $C^*$-algebra (see Section 3). We give a necessary and sufficient condition for the diagonalization. While the answer is negative in general, it is shown that...
positive normal functionals on the von Neumann algebra of $n \times n$ matrices over a semifinite countably decomposable von Neumann algebra can always be diagonalized.

Throughout the paper all $C^*$-algebras are assumed to be unital and $I$ will always denote the identity. If $L$ is a linear space and $n$ is a positive integer, $M_n(L)$ will denote the linear space of $n \times n$ matrices over $L$. If $\mathcal{H}$ is a Hilbert space and $x, y$ are two vectors in $\mathcal{H}$, we shall denote by $\omega_{x,y}$ the linear functional on $B(\mathcal{H})$, the algebra of all bounded operators on $\mathcal{H}$, given by $\omega_{x,y}(A) = \langle Ax,y \rangle$ ($A \in B(\mathcal{H})$). $\omega_x$ is the abbreviation of $\omega_{x,x}$.

The $n$-positive linear functionals of a $C^*$-algebra $\mathcal{A}$ form a positive cone that induces the natural order on $M_n(\mathcal{A})$. This order is also induced by the positive linear functionals on the $C^*$-algebra $M_n(\mathcal{A})$, as seen from the following well-known fact.

**Proposition 1.1.** The map $T : M_n(\mathcal{A}) \rightarrow (M_n(\mathcal{A}))^*$ given by

$$\left(T([\rho_{jk}])([A_{jk}]) = \sum_{j,k=1}^n \rho_{jk}(A_{jk}) \quad ([\rho_{jk}] \in M_n(\mathcal{A}), [A_{jk}] \in M_n(\mathcal{A}))\right)$$

is an order isomorphism between $M_n(\mathcal{A})$ and $(M_n(\mathcal{A}))^*$.

**Proof.** Each $\rho$ in $(M_n(\mathcal{A}))^*$ is uniquely determined by the linear functionals $\rho_{jk}$ on $\mathcal{A}$ given by $\rho_{jk}(A) = \rho(A \otimes E_{jk})$, where $A \otimes E_{jk}$ denotes the matrix in $M_n(\mathcal{A})$ whose $(j,k)$ entry is $A$ and all the other entries are 0; and we have:

$$\left(T([\rho_{jk}])([A_{jk}]) = \sum_{j,k=1}^n \rho_{jk}(A_{jk}) = \sum_{j,k=1}^n \rho(A_{jk} \otimes E_{jk}) = \rho([A_{jk}])\right)$$

If $[\rho_{jk}]$ is an $n$-positive linear functional on $\mathcal{A}$ and $[A_{jk}] \in M_n(\mathcal{A})^+$, then

$$\left(T([\rho_{jk}])([A_{jk}]) = \sum_{j,k=1}^n \rho_{jk}(A_{jk}) = \langle [\rho_{jk}(A_{jk})]e, e \rangle \geq 0,\right)$$

where $e$ is the vector $(1,1,\ldots,1)$ in $C^n$. Hence $T([\rho_{jk}])$ is positive. Conversely, if $\rho \in ((M_n(\mathcal{A}))^*)^+$, then $[\rho_{jk}] = T^{-1}(\rho)$ is an $n$-positive linear functional. Indeed, given $[A_{jk}]$ in $M_n(\mathcal{A})^+$ and $a = (a_1, \ldots, a_n) \in C^n$, the matrix $[\sum_{j,k=1}^n a_j a_k A_{jk}] = (\text{diag}(a_j)) [A_{jk}] (\text{diag}(a_j))$ belongs to $M_n(\mathcal{A})^+$; so that

$$0 \leq \rho([\sum_{j,k=1}^n a_j a_k A_{jk}]) = \sum_{j,k=1}^n a_j a_k \rho_{jk}(A_{jk}) = \langle [\rho_{jk}(A_{jk})]a, a \rangle.$$ 

Therefore $T$ is an order isomorphism.

**Remark.** Suppose $\mathcal{A}$ is a $C^*$-algebra acting on a Hilbert space $\mathcal{H}$. Then $M_n(\mathcal{A})$ is represented faithfully on $\oplus_{i=1}^n \mathcal{H}$, the direct sum of $n$ copies of $\mathcal{H}$, through the usual matrix action on “column vectors”; and we see that for each vector $x = (x_1, \ldots, x_n)$ in $\oplus_{k=1}^n \mathcal{H}$ the functional $\omega_x|_{M_n(\mathcal{A})}$ corresponds through the isomorphism $T$ to the $n$-positive linear functional $[\omega_{jk}]$, where $\omega_{jk} = \omega_{x_k,x_j}|_{\mathcal{A}}$ ($j, k \in \{1, \ldots, n\}$).
2. The Associated Representation

The following theorem establishes the analogue of the GNS representation for \( n \)-positive linear functionals. The construction involved is similar to the Stinespring construction for completely positive maps.

**Theorem 2.1.** If \( \mathcal{U} \) is a \( C^* \)-algebra and \( [\rho_{jk}] \) is an \( n \)-positive linear functional of \( \mathcal{U} \), then there is a representation \( \pi \) of \( \mathcal{U} \) on a Hilbert space \( \mathcal{H} \), and \( n \) vectors \( x_1, \ldots, x_n \) in \( \mathcal{H} \) such that \( \{x_1, \ldots, x_n\} \) is a generating set for \( \pi(\mathcal{U}) \) on \( \mathcal{H} \) and

\[
\rho_{jk} = \omega_{x_k x_j} \circ \pi \quad (j, k \in \{1, \ldots, n\}).
\]

**Proof.** Let \( \mathcal{L} \) be the linear space of all \( n \)-tuples of elements of \( \mathcal{U} \). Define the conjugate-bilinear form \( \langle \cdot, \cdot \rangle \) on \( \mathcal{L} \) by:

\[
\langle A, B \rangle = \sum_{j, k=1}^{n} \rho_{jk}(B^* A_k)(\overline{A} = (A_j), \overline{B} = (B_j)).
\]

We shall show that \( \langle \cdot, \cdot \rangle \) is an inner product on \( \mathcal{L} \). For this note first that \( \rho_{kj}(C^*) = \rho_{jk}(C) \) for each \( C \) in \( \mathcal{U} \) and \( j, k \in \{1, \ldots, n\} \). Indeed, this is apparent when \( j = k \). If, say, \( j < k \), consider the \( n \times n \) matrix \( M \) whose \((j, j) \) and \((j, k) \) entries are \( C^* \) and \( I \) respectively, and all the other entries are 0. The nonzero entries of \( M^* M \in \mathbb{M}_n(\mathbb{C})^+ \) form the matrix

\[
\begin{bmatrix}
CC^* & C \\
C^* & I
\end{bmatrix}.
\]

Consequently the matrix

\[
\begin{bmatrix}
\rho_{jj}(CC^*) & \rho_{jk}(C) \\
\rho_{kj}(C^*) & \rho_{kk}(I)
\end{bmatrix}
\]

is positive, and \( \rho_{kj}(C^*) = \rho_{jk}(C) \) for all \( C \) in \( \mathcal{U} \). Thus,

\[
\langle A, B \rangle = \sum_{j, k=1}^{n} \rho_{jk}(B^* A_k) = \sum_{j, k=1}^{n} \rho_{kj}(A^*_k B_j) = \langle \overline{B}, \overline{A} \rangle.
\]

Given \( \overline{A} = (A_j)^n \) in \( \mathcal{L} \), the matrix \( [A^*_j A_k] \) belongs to \( \mathbb{M}_n(\mathbb{C})^+ \), since it is the product

\[
\begin{bmatrix}
A_1 & \cdots & A_n
\end{bmatrix}^* \begin{bmatrix}
A_1 & \cdots & A_n
\end{bmatrix}.
\]

Hence the matrix \( \rho_{jk}(A^*_j A_k) \) is positive, and with \( e = (1, \ldots, 1) \in \mathbb{C}^n \) we have:

\[
0 \leq \langle \rho_{jk}(A^*_j A_k) e, e \rangle = \sum_{j, k=1}^{n} \rho_{jk}(A^*_j A_k) = \langle \overline{A}, \overline{A} \rangle.
\]

Therefore \( \langle \cdot, \cdot \rangle \) is an inner product on \( \mathcal{L} \). Let \( \mathcal{N} = \{A \in \mathcal{L} | \langle A, \overline{A} \rangle = 0\} \). By standard arguments the equation \( \langle A + \mathcal{N}, \overline{B} + \mathcal{N} \rangle = \langle \overline{A}, \overline{B} \rangle \) defines a definite inner product on the quotient space \( \mathcal{L}/\mathcal{N} \) whose completion is a Hilbert space \( \mathcal{H} \).

Each element \( A \) of \( \mathcal{U} \) defines the linear transformation \( \pi(A) \) on \( \mathcal{L}/\mathcal{N} \) by \( \pi(A)(A_j + \mathcal{N}) = (A A_j + \mathcal{N}) \). To see that \( \pi(A) \) is bounded note that \( R = \begin{bmatrix} A^* A & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \leq \begin{bmatrix} \|A\|^2 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} = S \).
so that with
\[ T = \begin{bmatrix} A_1 & \cdots & A_n \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 \end{bmatrix} \]
we have: \[ [A_j^* A_n A_k] = T^* R T \leq T^* S T = \| A \|^2 [A_j^* A_k], \]
and \[ \{\rho_{jk}(A_j^* A_n A_k)\} \leq \| A \|^2 \{\rho_{jk}(A_j^* A_k)\}, \]
by positivity of \[ \{\rho_{jk}\}. \]
Therefore
\[ \langle (A A_i), (A A_i) \rangle = \sum_{j,k=1}^{n} \rho_{jk}(A_j^* A_k A A_k) = \| A \|^2 \langle [\rho_{jk}(A_j^* A_k)] e, e \rangle \]
\[ \leq \| A \|^2 \langle [\rho_{jk}(A_j^* A_k)] e, e \rangle = \| A \|^2 \langle (A_i), (A_i) \rangle. \]

Consequently, \( \pi(A) \) is bounded on \( \mathcal{L}/\mathcal{M} \) and extends by continuity to a bounded linear operator on \( \mathcal{H} \) (also denoted by \( \pi(A) \)). It is easy to see that \( \pi \) is an algebra homomorphism of \( \mathcal{U} \). Furthermore, for any \( A = (A_i)_{i=1}^{n} \) and \( B = (B_i)_{i=1}^{n} \) in \( \mathcal{L} \),
\[ \langle (A)(A + M), (B + N) \rangle = \sum_{j,k=1}^{n} \rho_{jk}(A_j^* A_k A_k) = \sum_{j,k=1}^{n} \rho_{jk}((A^* B_j)^* A_k) \]
\[ = \langle (A + M), \pi(A^* B + N) \rangle, \]
which implies that \( \pi(A)^* = \pi(A^*) \). Therefore \( \pi \) is a representation of \( \mathcal{U} \) on \( \mathcal{H} \).

Finally, for each \( i \) in \( \{1, \ldots, n\} \) let \( x_i \) denote the vector \( \bar{I}_i + M \), where \( \bar{I}_i \) is the element of \( \mathcal{L} \) whose \( i \)-th component is \( 1 \) and all the other components are \( 0 \). Since \( \bar{A} + M = \sum_{i=1}^{n} \pi(A_i) x_i \) when \( \bar{A} = (A_i)_{i=1}^{n}, \{x_1, \ldots, x_n\} \) is a generating set for \( \pi(\mathcal{U}) \) on \( \mathcal{H} \). Also \( \rho_{jk}(A) = \langle \pi(A)(\bar{I}_k + M), (\bar{I}_k + N) \rangle = \langle \pi(A) x_k, x_j \rangle = \omega_{x_k, x_j}(\pi(A)) \) for all \( A \in \mathcal{U} \) and all \( j, k \in \{1, \ldots, n\} \). This completes the proof.

In the sequel the representation \( \pi \) of Theorem 2.1 will be called the representation engendered by \[ \{\rho_{jk}\}. \]

**Remark.** In the case when \[ \{\rho_{jk}\} \] is diagonal \( n \)-positive linear functional (that is, \( \rho_{jk} = 0 \) for \( j \neq k \) ) the representation engendered by \[ \{\rho_{jk}\} \] is (equivalent to) the \( n \)-fold direct sum of the GNS representations engendered by \( \rho_{jj} \) (\( j = 1, \ldots, n \)).

The following proposition provides a link between comparability of cyclic representations of \( \mathcal{U} \) and the order structure of \( \mathcal{M}_n(\mathcal{U}^*) \). Recall that two positive linear functionals of \( \mathcal{U} \) are called disjoint (quasi-equivalent) if the corresponding GNS representations are disjoint (quasi-equivalent). ([3, 10.31]).

**Proposition 2.2.** If \( \rho_{11} \) and \( \rho_{22} \) are positive linear functionals of a \( C^* \)-algebra \( \mathcal{U} \), then \( \rho_{11} \) and \( \rho_{22} \) are disjoint if and only if there are no nonzero linear functionals \( \rho_{12}, \rho_{21} = \rho_{12}^* \) on \( \mathcal{U} \) such that the matrix \[ \{\rho_{jk}\} \] (\( j, k = 1, 2 \)) is a \( 2 \)-positive linear functional.

**Proof.** Suppose \( \rho_{11} \) and \( \rho_{22} \) are disjoint and \[ \{\rho_{jk}\} \] is a \( 2 \)-positive linear functional of \( \mathcal{U} \). From Theorem 2.1 \( \rho_{jk}(A) = \langle \pi(A) x_k, x_j \rangle \) (\( A \in \mathcal{U}, j, k = 1, 2 \)),

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where \( \pi \) is the representation engendered by \( [\rho_{jk}] \) and \( x_1, x_2 \) are vectors in the Hilbert space of \( \pi \). Let \( P'_1 \) and \( P'_2 \) be the projections in \( \pi(\mathcal{U})' \) whose ranges are \( [\pi(\mathcal{U})x_1] \) and \( [\pi(\mathcal{U})x_2] \) respectively. Then the GNS representation engendered by \( \rho_{jj} \) \((j = 1, 2)\) is equivalent to the subrepresentation \( A \rightarrow \pi(A)P'_j \) \([3, 4.5.3]\). Since \( \rho_{11} \) and \( \rho_{22} \) are disjoint, the central carriers of projections \( P'_1 \) and \( P'_2 \) are orthogonal. In particular \( P'_1P'_2 = 0 \), so that \( 0 = \langle \pi(A)P'_2x_2, P'_1x_1 \rangle = \rho_{12}(A) = \rho_{21}(A^*) \) for each \( A \in \mathcal{U} \).

Conversely, suppose there are no nonzero functionals \( \rho_{12} \) and \( \rho_{21} \) such that \( [\rho_{jk}] \) is 2-positive. Letting \( \pi_1 \) and \( \pi_2 \) denote the representations of \( \mathcal{U} \) engendered by \( \rho_{11} \) and \( \rho_{22} \), consider the direct sum representation \( \Phi = \pi_1 \oplus \pi_2 \) on a Hilbert space \( \mathcal{H} \). Assuming, as we may, that \( \rho_{11}, \rho_{22} \neq 0 \), we have \( \pi_1(\mathcal{U}) = \Phi(\mathcal{U})E'_1, \pi_2(\mathcal{U}) = \Phi(\mathcal{U})E'_2 \) for some nonzero projections \( E'_1 \) and \( E'_2 \) in \( \Phi(\mathcal{U})' \) and \( \rho_{11} = \omega_{z_1} \circ \Phi, \rho_{22} = \omega_{z_2} \circ \Phi \) for some vectors \( z_1 \) and \( z_2 \) such that \( [\Phi(\mathcal{U})z_j] = E'_j(\mathcal{H}) \) \((j = 1, 2)\). If \( \pi_1 \) and \( \pi_2 \) are not disjoint, then there are nonzero projections \( F'_1 \leq E'_1, F'_2 \leq E'_2 \) and a partial isometry \( V' \) in \( \Phi(\mathcal{U})' \) such that \( V'^*V' = F'_1 \) and \( V'V'^* = F'_2 \). Since the projections \( F'_1 \) and \( F'_2 \) have ranges \( [\Phi(\mathcal{U})F'_1z_1] \) and \( [\Phi(\mathcal{U})F'_2z_2] \) respectively, the vectors \( y_1 = F'_1z_1 \) and \( y_2 = V'^*z_2 = V'F'_2z_2 \) are nonzero. Furthermore, as \( y_1 \) is a generating vector for the \( C^* \)-algebra \( \Phi(\mathcal{U})F'_1 \) acting on \( F'_1(\mathcal{H}) \), the functional \( \omega_{y_1,y_1} \circ \Phi \) is nonzero. The map \( T: M_2(\mathcal{U}) \rightarrow M_2(\Phi(\mathcal{U})F'_1) \), given by \( T([A_{jk}]) = [\Phi(A_{jk})F'_1] \) is a \( * \) homomorphism. Therefore the matrix \([\omega_{jk}]\), where \( \omega_{jk} = \omega_{y_j,y_j} \circ \Phi \) is a 2-positive linear functional on \( \mathcal{U} \). Letting \( \rho_{12} = \omega_{12} \) and \( \rho_{21} = \omega_{21} \), we have

\[
\begin{bmatrix}
\rho_{11} & \rho_{12} \\
\rho_{21} & \rho_{22}
\end{bmatrix} =
\begin{bmatrix}
\omega_{11} & \omega_{12} \\
\omega_{11} & \omega_{12}
\end{bmatrix} =
\begin{bmatrix}
\omega_{z_1} \circ \Phi & \omega_{y_1} \circ \Phi & 0 \\
0 & \omega_{z_2} \circ \Phi & \omega_{y_2} \circ \Phi
\end{bmatrix}.
\]

But

\[
\omega_{z_1} \circ \Phi - \omega_{y_1} \circ \Phi = \omega_{F'_1z_1} \circ \Phi + \omega_{(E'_1-F'_1)z_1} \circ \Phi - \omega_{F'_1z_1} \circ \Phi = \omega_{(E'_1-F'_1)z_1} \circ \Phi
\]

\[
\omega_{z_2} \circ \Phi - \omega_{y_2} \circ \Phi = \omega_{F'_2z_2} \circ \Phi + \omega_{(E'_2-F'_2)z_2} \circ \Phi - \omega_{F'_2z_2} \circ \Phi = \omega_{(E'_2-F'_2)z_2} \circ \Phi
\]

Thus \( [\rho_{jk}] \geq [\omega_{jk}] \geq 0 \), while \( \rho_{12} = \omega_{y_1,y_1} \circ \Phi \neq 0 \) contradicting the assumption. Consequently \( \rho_{11} \) and \( \rho_{22} \) are disjoint.

In the following proposition we characterize irreducible and factor representations engendered by \( n \)-positive linear functionals in terms of component functionals. For simplicity of exposition we consider the case of \( n \)-states. The general statement can be derived along the same lines.

**Proposition 2.3.** Let \( [\rho_{jk}] \) be an \( n \)-state of a \( C^* \)-algebra \( \mathcal{U} \) and \( \pi \) be the representation of \( \mathcal{U} \) engendered by \( [\rho_{jk}] \).

(i) \( \pi \) is irreducible if and only if \( \rho_{jj} \) \((j = 1, \ldots, n)\) are (equivalent) pure states and for each \( j \) and \( k \) there is a unitary \( U_{jk} \in \mathcal{U} \) such that \( \rho_{jk}(U_{jk}) = 1 \).
(ii) \( \pi \) is a factor representation if and only if \( \rho_{jj} \) \( (j = 1, \ldots, n) \) are quasi-equivalent factor states.

Proof. By Theorem 2.1 \( \rho_{jk} = \omega_{x_k, x_j} \circ \pi \) for some generating set of unit vectors \( x_1, \ldots, x_n \). As noted before, the GNS representation engendered by \( \rho_{jj} \) \( (j = 1, \ldots, n) \) is equivalent to the subrepresentation \( A \rightarrow \pi(A)P_j' \), where \( P_j' \) is the projection in \( \pi(\mathcal{Z}) \)' whose range is \( [\pi(\mathcal{Z})x_j] \). If \( \pi \) is irreducible, then \( P_j' = I \) and so \( \pi(\mathcal{Z}) = \pi(\mathcal{Z})P_j' \) for each \( j \). Thus the representations \( A \rightarrow \pi(A)P_j' \) are irreducible and \( \rho_{jj} \) are equivalent pure states. By transitivity of \( \pi(\mathcal{Z}) \), for each \( j \) and \( k \) there is a unitary \( U_{jk} \) in \( \mathcal{Z} \) such that \( \pi(U_{jk})x_k = x_j \) ([3,5.4.5]); so that \( \rho_{jk}(U_{jk}) = (\pi(U_{jk})x_k, x_j) = 1 \).

Assuming the converse we have \( (\pi(U_{jk})x_k, x_j) = \rho_{jk}(U_{jk}) = 1 \). As \( ||\pi(U_{jk})x_k|| = ||x_j|| = 1 \), it follows that \( \pi(U_{jk})x_k = x_j \). Therefore \( [\pi(\mathcal{Z})x_k] = [\pi(\mathcal{Z})x_k] \), and \( P_j' = P_k' \) for each \( j \) and \( k \). Since \( x_1, \ldots, x_n \) form a generating set for \( \mathcal{Z} \), this implies that \( P_j' = I \) for each \( j \). Consequently \( \pi(\mathcal{Z}) = \pi(\mathcal{Z})P_j' \) acts irreducibly.

If \( \pi(\mathcal{Z})'' \) is a factor, then the central carrier, \( C_{P_j'} \), of \( P_j' \) is the identity for each \( j \); so that \( \pi(\mathcal{Z}) \) is isomorphic to \( \pi(\mathcal{Z})P_j' \), and \( \rho_{jj} \) are quasi-equivalent factor states ([3, 10.3.3]).

Conversely, if \( \rho_{jj} \) are quasi-equivalent factor states, then \( \pi(\mathcal{Z})''P_j' \) is a factor and \( C_{P_j'} = I \) for each \( j \). If \( C \) is a nonzero projection in the center of \( \pi(\mathcal{Z})'' \), then \( CP'_i \neq 0 \) for some \( i \). Since \( CP'_i \) belongs to the center of \( \pi(\mathcal{Z})''P_j' \), \( CP'_i = P'_i \); so that \( C \geq C_{P_i'} = I \). Hence \( C = I \), and \( \pi(\mathcal{Z})'' \) is a factor.

Proposition 2.4. Let \( [\rho_{jk}] \) and \( [\sigma_{jk}] \) be \( n \)-positive linear functionals on a \( C^\ast \)-algebra \( \mathcal{Z} \), and \( \pi \) be the representation engendered by \( [\rho_{jk}] \) on a Hilbert space \( \mathcal{H} \), so that \( \rho_{jk} = \omega_{x_k, x_j} \circ \pi \) for some generating set of vectors \( x_1, \ldots, x_n \) for \( \pi(\mathcal{Z}) \). If \( [\sigma_{jk}] \leq [\rho_{jk}] \), then there is a positive operator \( H' \) in the unit ball of \( \pi(\mathcal{Z})' \) such that \( \sigma_{jk}(A) = \omega_{x_k, x_j}(H'\pi(A)) \) \( (A \in \mathcal{Z}; \ j, k = 1, \ldots, n) \).

Proof. Let \( S \) denote the linear span of the set \( \{\pi(\mathcal{Z})x_i|i = 1, \ldots, n\} \). Define a conjugate-bilinear form \( \psi \) on \( S \) by:

\[
\psi(x, y) = \sum_{j,k=1}^{n} \sigma_{jk}(B_j^*A_k),
\]
where \( x = \sum_{i=1}^{n} \pi(A_i)x_i \) and \( y = \sum_{i=1}^{n} \pi(B_i)x_i \). Applying the Cauchy-Schwarz inequality,

\[
|\psi(x,y)|^2 = \left| \sum_{j,k=1}^{n} \sigma_{jk}(B^*_jA_k) \right|^2 \leq \sum_{j,k=1}^{n} \sigma_{jk}(A^*_jA_k) \sum_{j,k=1}^{n} \sigma_{jk}(B^*_jB_k)
\]

\[
\leq \sum_{j,k=1}^{n} \rho_{jk}(A^*_jA_k) \sum_{j,k=1}^{n} \rho_{jk}(B^*_jB_k) = \|x\|^2\|y\|^2,
\]

since \( [\sigma_{jk}] \leq [\rho_{jk}] \). Thus \( \psi \) is bounded by 1 on \( \mathcal{H} \) and extends by continuity to a conjugate-bilinear form on \( \mathcal{H} \) (also denoted by \( \psi \)). Therefore there is a positive operator \( H' \) of norm 1 on \( \mathcal{H} \) such that \( \psi(x,y) = \langle H'x, y \rangle \) for all \( x \) and \( y \) in \( \mathcal{H} \). Since

\[
\langle H'\pi(A)\pi(B)x_k, \pi(C)x_j \rangle = \psi(\pi(AB)x_k, \pi(C)x_j) = \sigma_{jk}(C^*AB)
\]

\[
= \sigma_{jk}((A^*C)^*B) = \psi(\pi(B)x_k, \pi(A^*C)x_j)
\]

\[
= \langle \pi(A)H'\pi(B)x_k, \pi(C)x_j \rangle
\]

for all \( A, B, C \) in \( \mathcal{U} \) and \( x_1, \ldots, x_n \) is a generating set, it follows that \( H' \in \pi(\mathcal{Z})' \). Also we have \( \sigma_{jk}(A) = \psi(\pi(A)x_k, x_j) = \langle H'\pi(A)x_k, x_j \rangle \).

The set of all \( n \)-states of a \( C^* \)-algebra \( \mathcal{U} \) is convex and compact in the topology of weak \(^*\) convergence on \( M_n(\mathcal{Z}^*) \). From the Krein-Milman theorem it is the weak \(^*\) closure of the convex hull of extremal \( n \)-states.

**Proposition 2.5.** Let \( [\rho_{jk}] \) be an \( n \)-state of a \( C^* \)-algebra \( \mathcal{U} \). Suppose \( \rho_{jj} \) (\( j = 1, \ldots, n \)) are pure states of \( \mathcal{U} \) and whenever \( \rho_{ss} \) is equivalent to \( \rho_{tt} \) for some \( s \) and \( t \) in \( \{1, \ldots, n\} \), there is a unitary \( U_{st} \) in \( \mathcal{U} \) such that \( \rho_{st}(U_{st}) = 1 \). Then \( [\rho_{jk}] \) is an extremal \( n \)-state.

**Proof.** Let \( m \) be the number of equivalence classes of the pure states \( \rho_{jj} \). By Proposition 2.2 the matrix \( [\rho_{jk}] \) is a block-diagonal sum of the matrices \( [\rho_{j(i),k(i)}] \) \( (1 \leq i \leq m) \) corresponding to each equivalence class; and for each \( i \) \( [\rho_{j(i),k(i)}] \) is an \( n_i \)-state for some \( n_i \). By Proposition 2.3(i) the representation \( \pi_i \) engendered by \( [\rho_{j(i),k(i)}] \) is irreducible for each \( i \), and is equivalent to the GNS representation induced by any one of the states \( \rho_{j(i),j(i)} \). Since \( \pi_i \) is (equivalent to) the subrepresentation of the representation \( \pi \) engendered by \( [\rho_{jk}] \), \( \pi \) is the sum of disjoint representations \( \pi_i \). Let \( C_i \) denote the central support of \( \pi_i \) (so that \( \pi_i(\mathcal{Z}) = \pi(\mathcal{Z})C_i \)).

Suppose \( a \cdot [\sigma_{jk}] \leq [\rho_{jk}] \) for some \( n \)-state \( [\sigma_{jk}] \) and \( 0 < a \leq 1 \). Then \( a \cdot \sigma_{jk}(A) = \omega_{x_k,x_j}(H^*\pi(A)) \) (\( A \in \mathcal{U} \)) for some \( H' \in \pi(\mathcal{Z})' \), from Proposition 2.4. But, since \( H'C_i \) belongs to \( \pi_i(\mathcal{Z})' \) and \( H'C_i \neq 0 \), \( H'C_i = a \cdot C_i \) for each \( i \). Consequently \( H'C_i = aI \), \( [\sigma_{jk}] = [\rho_{jk}] \), and \( [\rho_{jk}] \) is extremal.
Proposition 2.6. There is a one-to-one correspondence between the set of all \(n\)-positive linear functionals \(\rho = [\rho_{jk}]\) of a C*-algebra \(\mathcal{U}\) and the set of all completely positive maps \(\Psi: \mathcal{U} \to M_n(\mathbb{C})\), given by \(\Psi_\rho(A) = [\rho_{jk}(A)]\) \((\rho = [\rho_{jk}], A \in \mathcal{U})\).

Proof. First we show that \(\Psi_\rho\) is completely positive. Let \([A_{im}] = \sum_{i,m=1}^q A_{im} \otimes F_{im}\) be a positive element of \(M_q(\mathcal{U})\) for some integer \(q\). Since \([A_{im}]\) is a sum of positive elements of \(M_q(\mathcal{U})\) of the form \([B_i^* B_m]\), we may assume that \([A_{im}] = \sum_{i,m=1}^q A_{im} \otimes F_{im}\). Moreover, by Theorem 2.1 we may assume that \(\mathcal{U}\) acts on a Hilbert space \(\mathcal{H}\) and \(\rho_{jk} = \omega_{x_k x_j}\) for some vectors \(x_1, \ldots, x_n\) in \(\mathcal{H}\). Then \([\Psi_\rho(A_{im})]\) is the matrix

\[
\sum_{i,m=1}^q \left( \sum_{j,k=1}^n (A_{im} x_k, A_{ij} x_j) \otimes E_{jk} \right) \otimes F_{im} \quad \text{in} \quad M_q(M_n(\mathbb{C})).
\]

This matrix can be identified with the representing matrix of the operator \(T^* T\) in some basis \(\{u_s\} (s = 1, \ldots, qn)\) for a \(qn\)-dimensional subspace \(\mathcal{H}_0\) of \(\mathcal{H}\) containing all the vectors \(A_{im} x_k\) \((m = 1, \ldots, q; k = 1, \ldots, n)\), where \(T\) is the operator on \(\mathcal{H}_0\) given by \(Tu_s = A_{im} x_k\) when \(s = (m-1)n + k\). Consequently \([\Psi_\rho(A_{im})] \in M_q(M_n(\mathbb{C}))^+\), and \(\Psi_\rho\) is completely positive.

On the other hand, given a completely positive map \(\Psi\) from \(\mathcal{U}\) into \(M_n(\mathbb{C})\), let \(\rho_{jk}\) be the linear functional on \(\mathcal{U}\) given by \(\rho_{jk}(A) = (\Psi(A))_{jk}\), where \((\Psi(A))_{jk}\) denotes the \((j,k)\) entry of the matrix \(\Psi(A)\). Given \([B_{jk}]\) in \(M_n(\mathcal{U})^+\), the matrix \(S = \sum_{j,k=1}^n \Psi(B_{jk}) \otimes F_{jk}\) belongs to \(M_n(M_n(\mathbb{C}))^+\) by the complete positivity of \(\Psi\). Thus, with \(\{e_i\}\) the standard basis for \(\mathbb{C}^n\) and \(z = \oplus_{i=1}^n e_i\) in \(\oplus_{i=1}^n \mathbb{C}^n\), we have \(0 \leq (Sz, z) = \sum_{j,k=1}^n \rho_{jk}(B_{jk})\). From Proposition 1.1 it now follows that \([\rho_{jk}]\) \((= \rho)\) is an \(n\)-positive linear functional, and it is clear that \(\Psi = \Psi_\rho\).

The following corollary is an immediate consequence of Propositions 2.5 and 2.6.

Corollary 2.7. Let \([\rho_{jk}]\) be an \(n\)-state of a C*-algebra \(\mathcal{U}\). If \([\rho_{jk}]\) satisfies the assumptions of Proposition 2.5 and \(\rho_{jk}(I) = 0\) for all \(j \neq k\), then the map \(\Psi: \mathcal{U} \to M_n(\mathbb{C})\) given by \(\Psi_\rho(A) = [\rho_{jk}(A)]\) is an extreme point in the set of all identity preserving completely positive maps from \(\mathcal{U}\) into \(M_n(\mathbb{C})\).

3. Unitary diagonalization

This section is concerned with the problem of diagonalization of linear functionals on a matrix algebra over a C*-algebra. Given a (unital) C*-algebra \(\mathcal{U}\), we shall denote by \(E_i\) \((i = 1, \ldots, n)\) the projection in \(M_n(\mathcal{U})\) whose matrix has \(I\) on the \(i\)th diagonal entry and has all the other entries 0. A linear functional \(\varphi\) on \(M_n(\mathcal{U})\) is called diagonal if \(\varphi(E_j A E_k) = 0\) for all \(j \neq k\) and \(A \in M_n(\mathcal{U})\).
This is equivalent to the condition that \([\varphi_{jk}]\) is diagonal. The problem we consider is whether for a linear functional \(\omega\) on \(M_n(\mathcal{B})\) there exist a unitary \(U \in M_n(\mathcal{B})\) such that the functional \(\varphi\) given by \(\varphi(A) = \omega(UAU^*)\) is diagonal. In this case we shall say that \(\omega\) is unitarily diagonalizable. Recall that the centralizer of a linear functional \(\omega\) on a \(C^*\)-algebra \(\mathcal{B}\) is the set \(\mathcal{C}_\omega = \{H \in \mathcal{B} | \omega(AH) = \omega(HA)\text{ for all } A \in \mathcal{B}\}\). \(\mathcal{C}_\omega\) is a \(C^*\)-subalgebra of \(\mathcal{B}\) if \(\omega\) is positive.

**Proposition 3.1.** Let \(\mathcal{B}\) and \(E_i (i = 1, \ldots, n)\) be as above. A linear functional \(\omega\) on \(M_n(\mathcal{B})\) is unitarily diagonalizable if and only if there are \(n\) orthogonal projections \(F_i\) \((i = 1, \ldots, n)\) with sum \(I\) in the centralizer \(\mathcal{C}_\omega\), such that \(E_i\) is (Murray-von Neumann) equivalent to \(F_i\) in \(M_n(\mathcal{B})\).

**Proof.** If such a family \(F_i\) \((i = 1, \ldots, n)\) exists, let \(V_i\) be the partial isometry such that \(V_iF_iV_i^* = E_i\). The element \(U = \sum_{i=1}^n V_i\) is a unitary and \(UF_iU^* = E_i\). Therefore for each \(A \in M_n(\mathcal{B})\) and \(j \neq k\) we have:

\[
\omega(UF_jAF_kU^*) = \omega(UF_jU^*UAU^*UE_kU^*) = \omega(F_jUAU^*F_k) = 0;
\]

so that \(\omega\) is unitarily diagonalizable.

Conversely, if there exists a unitary \(U\) in \(M_n(\mathcal{B})\) such that \(\omega(UF_jAF_kU^*) = 0\) for all \(j \neq k\) and \(A \in M_n(\mathcal{B})\), let \(F_i = UE_iU^*\) \((i = 1, \ldots, n)\). Since \(\sum_{i=1}^n F_i = I\) and \(F_jF_k = 0\) for all \(j \neq k\), we have:

\[
\omega(AF_i) = \omega \left( \sum_{j,k=1}^n F_jAF_kF_i \right) = \sum_{j=1}^n \omega(F_jAF_i) = \sum_{j=1}^n \omega(UF_jU^* AUE_jU^*) = \omega(UF_iU^* AUE_iU^*) = \omega(F_iAF_i);
\]

Similarly \(\omega(F_iA) = \omega(F_iAF_i)\) for each \(i\) and \(A\). Thus, the family \(F_i\) \((i = 1, \ldots, n)\) belongs to \(\mathcal{C}_\omega\) and has the asserted properties.

**Theorem 3.2.** If \(\mathcal{B}\) is a semifinite countably decomposable von Neumann algebra and \(\omega\) is a positive normal linear functional on \(M_n(\mathcal{B})\), then \(\omega\) is unitarily diagonalizable.

**Proof.** The von Neumann algebra \(M_n(\mathcal{B})\) is semifinite and countably decomposable. Hence it admits a faithful normal semifinite tracial weight \(\tau\). From the version of Radon-Nikodym theorem in [3, 9.2.19] there is a positive operator \(K\) in the unit ball of \(M_n(\mathcal{B})\) such that \(\tau(I - K) < \infty\) and

\[
(\star) \quad \tau((I - K)A) = \omega(KA) = \omega(AK) \quad \text{for all } A \in M_n(\mathcal{B}).
\]

We shall show first that the relative commutant \(\{K\}^\prime = \{K\}^\prime \cap M_n(\mathcal{B})\) belongs to the centralizer \(\mathcal{C}_\omega\). For this we note that \(K\) is a one-to-one map. Indeed, if \(P\) is the null-projection of \(K\), then \(KP = 0\) and from \((\star)\), \(\tau(P) = \tau((I - K)P) = \omega(KP) = 0\). Since \(\tau\) is faithful, this implies \(P = 0\). Thus, the inverse \(K^{-1}\) is a closed densely defined positive operator affiliated with the
abelian von Neumann algebra $\mathcal{M}_0$ generated by $K$ ([3, 5.6.12]). If $E_n$ denotes the spectral projection of $K^{-1}$ corresponding to the interval $[1/n, n]$, then $E_n \to I$ strongly and $K^{-1}E_n$ belongs to $\mathcal{M}_0$. Consequently if $H \in \{K\}^c$, then $\omega(AH) = \lim_n \omega(KK^{-1}E_nAH) = \lim_n \tau((I - K)K^{-1}E_nAH) = \lim_n \tau(H(I - K)K^{-1}E_nA) = \lim_n \omega(E_nAH) = \lim_n \omega(E_nAH) = \omega(HA)$, from (*). Hence $\{K\}^c \subseteq \mathcal{C}_\omega$. In particular, if $\mathcal{M}$ is a maximal abelian subalgebra of $M_n(\mathcal{R})$ containing $K$, then $\mathcal{M} \subseteq \mathcal{C}_\omega$. From [2, Theorem 3.18] $\mathcal{M}$ contains $n$ orthogonal equivalent projections $F_i$ ($i = 1, \ldots, n$) with sum $I$. By comparison theory $F_i$ is equivalent to $E_i$ for each $i$. Therefore $\omega$ is unitarily diagonalizable, by Proposition 3.1.

Remark. The possibility of unitary diagonalization of normal states on matrix algebras over semi-finite countably decomposable von Neumann algebras is due to the fact that the centralizers are always relatively large. This need not be the case if the von Neumann algebra is of type III. In [1] the example was shown of a faithful normal state on a type III factor, whose centralizer consists of the scalar multiples of the identity. This indicates a counterexample illustrating the failure of unitary diagonalization in the type III case.

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