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COMPOSITION OPERATORS INDUCED BY FUNCTIONS WITH SUPREMUM STRICTLY SMALLER THAN 1

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ABSTRACT. We give some partial solutions to the following problem. Does a function analytic in the unit disc D with supremum strictly smaller than 1, induce a bounded composition operator on all weighted Hardy spaces $H^2(\beta)$.

For H a Hilbert space of functions on a set X, and ϕ a function that maps X into itself we define the composition operator C_ϕ on H by $C_\phi f = f \circ \phi$.

Let $\beta = \{\beta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\beta_0 = 1$ and

$$\lim_{n\to\infty} \frac{\beta_{n+1}}{\beta_n} = 1.$$

Define the set $H^2(\beta)$ to be the set of all complex formal power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty$. Then $H^2(\beta)$ is a Hilbert space of functions analytic in the unit disc D with the inner product

$$(f, g)_\beta = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta_n^2$$

for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ (for details see [11]). If $\beta_n = 1$ for all n, then $H^2(\beta)$ is the classical Hardy space H^2, and some general properties of composition operators on H^2 are known (see for example [8, 10, 4, 2, and 5]). There are few results in the case of more general $H^2(\beta)$ spaces (see for example [1, 6, 3, 9, and 12]) and it is interesting to see how they can differ from the "H^2 case," or to see how difficult even the basic question about boundedness of composition operators can become.

We can hope to get some results about boundedness of composition operators on general $H^2(\beta)$ spaces if they are induced by some special types of functions. Some interesting examples are the functions that have sup norm strictly smaller than 1. In the "H^2 case" it is not too hard to see that they induce trace class

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composition operators (see [10]) and we shall see below that getting such a nice result is not accidental.

For the definitions of trace class and Schatten p-class for $p > 0$ see [7, Chapter III].

Let $\|\phi\|_\infty$ denote the supremum of $\{\phi(z) \colon z \in D\}$.

Proposition 1. *Let the space $H^2(\beta)$ be such that every function in $H^2(\beta)$ with sup norm strictly smaller than 1 induces a bounded composition operator on $H^2(\beta)$. Then, if $\phi \in H^2(\beta)$ and $\|\phi\|_\infty < 1$, C_ϕ is in every Schatten p-class, $p > 0$, of $H^2(\beta)$.*

Proof. Let $\|\phi\|_\infty = r < 1$ and $r < r_1 < 1$. Let $\psi(z) = r_1 z$. We shall use the fact that for every $H^2(\beta)$ space $\psi \in H^2(\beta)$. Also, C_ψ is in every Schatten p-class of $H^2(\beta)$ because

$$C_\Psi \frac{z^n}{\beta_n} = \frac{\Psi^n}{\beta_n} = r_1^n \frac{z^n}{\beta_n}$$

and for $p > 0$

$$\sum_{n=0}^{\infty} r_1^{pn} < \infty.$$

Let $\phi_1 = 1/r_1 \cdot \phi$. Then ϕ_1 belongs to the given space $H^2(\beta)$ and $\|\phi_1\|_\infty = (1/r_1) \cdot \|\phi\|_\infty = r/r_1 < 1$. So the operator C_{ϕ_1} is bounded on $H^2(\beta)$ and because $\phi = \psi \circ \phi_1$, we have $C_\phi = C_{\phi_1} \cdot C_\psi$. But C_{ϕ_1} is bounded, C_ψ is in every Schatten p-class of $H^2(\beta)$ and so C_ϕ is also in every Schatten p-class of $H^2(\beta)$. □

We can distinguish two different types of spaces $H^2(\beta)$: one when the sequence β is bounded and the other when the sequence β is unbounded. The following proposition shows an interesting correlation between the space $H^2(\beta)$ and the space H^∞ (the space of functions bounded and analytic in the unit disc) in each of the cases.

Proposition 2. *$H^\infty \subset H^2(\beta)$ if and only if the sequence β is bounded.*

Proof. Suppose that there exists an $M > 0$ such that $\beta_n \leq M$ for all n, and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If $f \in H^2$ then $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, and

$$\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \leq M^2 \sum_{n=0}^{\infty} |a_n|^2 < \infty,$$

i.e. $f \in H^2(\beta)$. Thus $H^\infty \subset H^2 \subset H^2(\beta)$.

Suppose now that the sequence β is unbounded. We are going to show that then there is a function $\phi \in H^\infty$ with $\phi \notin H^2(\beta)$. Let the subsequence

$\{\beta_{n_k}\}_{k=1}^{\infty}$ be such that $\beta_{n_k} \geq k$ and $n_{k+1} > n_k$, $k \geq 1$ and let $1 < b < \frac{3}{2}$. Let

$$c = \sum_{n=1}^{\infty} \frac{1}{k^b} \quad \text{and} \quad f(z) = \frac{1}{2c} \sum_{n=0}^{\infty} a_n z^n$$

where

$$a_n = \begin{cases} \frac{1}{k^b}, & n = n_k, \\ 0, & n \neq n_k. \end{cases}$$

Then

$$\|f\|_{\infty} \leq \frac{1}{2c} \sum_{k=1}^{\infty} \frac{1}{k^b} = \frac{1}{2}$$

and

$$\begin{aligned} \|f\|_{\beta}^2 &= \frac{1}{4c^2} \sum_{k=1}^{\infty} \frac{1}{k^{2b}} \cdot \beta_{n_k}^2 \\ &\geq \frac{1}{4c^2} \sum_{k=1}^{\infty} \frac{1}{k^{2b}} k^2 = \frac{1}{4c^2} \sum_{k=1}^{\infty} \frac{1}{k^{2(b-1)}} = \infty, \end{aligned}$$

since $0 < 2(b-1) < 1$. So $f \in H^{\infty}$, and $f \notin H^2(\beta)$. □

Note that in the case when the sequence β is bounded and 0 is not an accumulation point for β, then the β-norm is actually equivalent to the H^2-norm and so $H^2(\beta) = H^2$.

Also note that if the sequence β is bounded by M and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f \in H^{\infty}$, then

$$\|f\|_{\beta}^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \leq M^2 \sum_{n=0}^{\infty} |a_n|^2 = M^2 \|f\|_2^2 \leq M^2 \|f\|_{\infty}^2.$$

Now, we have the following result.

Theorem 1. *If $H^{\infty} \subset H^2(\beta)$ and $\|\phi\|_{\infty} < 1$, then C_{ϕ} is in every Schatten p-class, $p > 0$, of $H^2(\beta)$.*

Proof. Using Proposition 1 the only thing that we have to show is that if $H^{\infty} \subset H^2(\beta)$, then C_{ϕ} is bounded on $H^2(\beta)$ for any ϕ with $\|\phi\|_{\infty} < 1$. This is rather trivial because for any f in $H^2(\beta)$ we actually have that f is continuous on $\overline{\phi(D)} \subset D$, i.e., $f \circ \phi \in H^{\infty} \subset H^2(\beta)$, and this implies that C_{ϕ} is bounded on $H^2(\beta)$. □

There is one special case when the sequence β is unbounded for which we can prove that every function ϕ in $H^2(\beta)$ with sup norm strictly smaller than 1 induces a bounded composition operator on $H^2(\beta)$, i.e., an operator in every Schatten p-class of $H^2(\beta)$. For that, we need the following.

For a given space $H^2(\beta)$, we define $H^{\infty}(\beta)$ to be the set

$$\{f \colon fg \in H^2(\beta), \forall g \in H^2(\beta)\}.$$

Because the function $e_0(z) = 1$ for $z \in D$ belongs to $H^2(\beta)$, we have that $H^\infty(\beta) \subset H^2(\beta)$. We define an " ∞, β " norm on $H^\infty(\beta)$, using multiplication operators on $H^2(\beta)$. For ϕ in $H^\infty(\beta)$, we have

$$\|\phi\|_{\infty,\beta} = \|M_\phi\|$$

where M_ϕ is the multiplication operator on $H^2(\beta)$. In the case of the Hardy space H^2, the space $H^\infty(\beta)$ is H^∞, the space of bounded analytic functions on the unit disc D, and the ∞, β-norm of H^∞ is the usual sup norm.

In some special cases (as for example the spaces S_a when $a > \frac{1}{2}$ and Q_a when $0 < a < 1$) the β norm of $H^2(\beta)$ and ∞, β-norm of $H^\infty(\beta)$ are equivalent, i.e. $H^\infty(\beta) = H^2(\beta)$. Then we say that the space $H^2(\beta)$ is strictly cyclic. For details see [11].

Theorem 2. *Let the sequence β be such that $H^2(\beta) = H^\infty(\beta)$, and the function ϕ in $H^2(\beta)$ be such that $\|\phi\|_\infty < 1$. Then the operator C_ϕ is in every Schatten p-class, $p > 0$, of $H^2(\beta)$.*

Proof. Let $H^2(\beta) = H^\infty(\beta)$. By Corollary 1 to Proposition 31 in [11], we have that the spectrum of each element of the Banach algebra $H^\infty(\beta)$ is its range on $\overline{D}$. By Proposition 20 in [11], the spectrum of ϕ as an element of $H^\infty(\beta)$ is the same as the spectrum of the multiplication operator M_ϕ on $H^2(\beta)$.

Suppose now that $\phi \in H^2(\beta)$ and $\|\phi\|_\infty < 1$. Then for the spectral radius $r(M_\phi)$ we have

$$r(M_\phi) = \lim_{n\to\infty} \|M_\phi^n\|^{1/n} < 1.$$

We use this fact to prove first that C_ϕ is in the trace class of $H^2(\beta)$. Let $f_n = z^n/\beta_n$, where $n \geq 0$. The sequence $\{f_n\}$ is orthonormal in $H^2(\beta)$, and

$$\sum_{n=0}^{\infty} \|C_\infty f_n\| = \sum_{n=0}^{\infty} \frac{\|\phi^n\|_\beta}{\beta_n}.$$

For any f in $H^\infty(\beta)$, we have that

$$\|f\|_\beta = \|M_f e_0\|_\beta \leq \|M_f\| \, \|e_0\|_\beta = \|M_f\|$$

and so $\|f\|_\beta \leq \|f\|_{\infty,\beta}$. But then

$$\sum_{n=0}^{\infty} \|C_\phi f_n\| \leq \sum_{n=0}^{\infty} \frac{\|\phi^n\|_{\infty,\beta}}{\beta_n} = \sum_{n=0}^{\infty} \frac{\|M_\phi^n\|}{\beta_n} < \infty$$

because

$$\lim_{n\to\infty} \left(\frac{\|M_\phi^n\|}{\beta_n}\right)^{1/n} < \lim_{n\to\infty} \left(\frac{1}{\beta_n}\right)^{1/n} = 1.$$

So every function in $H^2(\beta)$ with sup norm strictly smaller than 1 induces a composition operator that is in the trace class of $H^2(\beta)$ and by Proposition 1 it induces a composition operator that is in every Schatten p-class of $H^2(\beta)$. □

An interesting result that we would like to mention is a consequence of a more general consideration in [9]. If the sequence β is such that the functions in $H^2(\beta)$ are continuous on $\overline{D}$ and disc automorphisms induce bounded composition operators, then C_ϕ compact on $H^2(\beta)$ implies that $\|\phi\|_\infty < 1$ (as for example in the case of spaces S_a, $a > \frac{1}{2}$).

Placing more restrictions on the function ϕ, we can get another result. Let $A_{\overline{D}} = \{f : f \text{ is analytic in some neighborhood of } \overline{D}\}$.

Theorem 3. *Let $\phi \in A_{\overline{D}}$ and $\|\phi\|_\infty < 1$. Then the operator C_ϕ is bounded on all $H^2(\beta)$ spaces.*

Proof. If $\|\phi\|_\infty < 1$, then there exists a disc $D_1 \supset \overline{D}$ such that $\phi(D_1) \subset D$, since $\phi \in A_{\overline{D}}$. But then for f in $H^2(\beta)$, $f \circ \phi$ is analytic on D_1, i.e., $f \circ \phi \in A_{\overline{D}}$. The result follows since $A_{\overline{D}} \subset H^2(\beta)$. □

Corollary 1. *If $\phi \in A_{\overline{D}}$ and $\|\phi\|_\infty < 1$, then C_ϕ is in every Schatten p-class, $p > 0$, of all $H^2(\beta)$ spaces.*

Proof. Let $\|\phi\|_\infty = r < 1$ and $r < r_1 < 1$. Then the function $\psi(z) = r_1 z$ induces a composition operator that is in every Schatten p-class of any $H^2(\beta)$. The function $\phi_1 = (1/r_1)\phi$ belongs to $A_{\overline{D}}$ because ϕ does, and also $\|\phi_1\|_\infty = r/r_1 < 1$. By Theorem 3 the operator C_{ϕ_1} is bounded on all $H^2(\beta)$ spaces, and the same idea as in the proof of Proposition 1 leads us to the conclusion that $C_\phi = C_{\phi_1} \cdot C_\psi$ is in every Schatten p-class of any $H^2(\beta)$. □

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