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MOSCO CONVERGENCE AND THE KADEC PROPERTY

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ABSTRACT. We study the relationship between Wijsman convergence and Mosco convergence for sequences of convex sets. Our central result is that Mosco convergence and Wijsman convergence coincide for sequences of convex sets if and only if the underlying space is reflexive with the dual norm having the Kadec property.

1. INTRODUCTION

There are many notions of convergence of sequences of convex sets in a normed space. This paper was stimulated by the analysis in Beer [Be] where various notions of set convergence are characterized when the sets in question are hyperplanes. One of the most fruitful notions is that of Mosco convergence [At, Be, Mo1, Mo2, Sa-We, So, Ts]. Mosco convergence is particularly useful in reflexive spaces since then polarity is sequentially bicontinuous (E_n converges to E if and only if E_n^0 converges to E^0; see [Mo2]). Another more intuitive notion is that of Wijsman convergence [Wi]. It has been known for some time that these notions coincide when the underlying space is reflexive and the norm used is Fréchet differentiable [At, So, Ts] and when the space is finite dimensional. Beer has shown that this coincidence fails, even for sequences of hyperplanes, whenever the dual norm is not Kadec.

Our central result (Theorem 3.4) is that Mosco convergence and Wijsman convergence coincide for sequences of convex sets if and only if the underlying space is reflexive with the dual norm having the Kadec property. This recaptures the results mentioned above. We establish this by providing a circuit of characterizations of reflexive spaces whose dual norm is Kadec (Theorem 3.1). The proof of this result is motivated both by Tsukada's technique and Beer's analysis. In §2 we make the appropriate definitions and provide some preliminary results.

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2. Preliminary results

Throughout this paper X is a real Banach space endowed with a fixed norm, $\|\ \|$.

A sequence of sets $\{C_n\}$ in X is said to *converge Wijsman* to a set C if

$$\lim_{n\to\infty} d(x, C_n) = d(x, C)$$

for each x in X, where $d(x, C) := \inf_{c\in C} \|x - c\|$ and $d(x, \varnothing) = \infty$. We will write $\lim_W C_n = C$.

A sequence of sets $\{C_n\}$ in X is said to *converge Mosco* to a set C if

M(i): for each $x \in C$ there exist, for large n, $x_n \in C_n$ such that x_n converges in norm to x.

M(ii): if there is a subsequence n' with $x_{n'} \in C_{n'}$ such that $x_{n'}$ converges weakly to a point x then $x \in C$.

We will write $\lim_M C_n = C$.

It is apparent that Wijsman convergence depends on the precise norm used, while Mosco convergence is preserved by equivalent renorming. We now record some useful relationships.

Proposition 2.1. (a) *If* C_n *are subsets of* X *and* $\lim_W C_n = \varnothing$ *then* $\lim_M C_n = \varnothing$.

(b) *Conversely, if* X *is reflexive and* $\lim_M C_n = \varnothing$ *then* $\lim_W C_n = \varnothing$.

Proof. (a) For C empty, M(i) holds vacuously. If M(ii) fails with C empty then there is a subsequence n' with $x_{n'} \in C_{n'}$ such that $x_{n'}$ converges weakly to a point x. But now $\{x_{n'}\}$ is bounded and $\liminf d(x, C_n) < \infty$ whence $\lim_W C_n = \varnothing$.

(b) If $\liminf d(x, C_n) < \infty$ we can find a bounded subsequence $c_{n'} \in C_{n'}$. By weak compactness there is a further subsequence $c_{n''} \in C_{n''}$ with weak limit x_0. By M(ii) we see that $x_0 \in \lim_M C_n = \varnothing$. This contradiction establishes that $\liminf d(x, C_n) = \infty$ so that $\lim_W C_n = \varnothing$. □

Theorem 2.2. (a) *If* X *is reflexive and* $\lim_M C_n = C$ *then* $\lim_W C_n = C$.

(b) *If* X *is nonreflexive there is a sequence* $\{C_n\}$ *of compact convex sets with* $\lim_M C_n = C$ *compact convex and nonempty, but* $\lim_W C_n$ *fails to exist.*

Proof. (a) We may suppose that C is nonempty. Let $\varepsilon > 0$. Choose $c \in C$ so that $\|x - c\| \le d(x, C) + \varepsilon$. By M(i) select $c_n \in C_n$ with c_n converging in norm to c. Then

$$\limsup d(x, C_n) \le \lim \|x - c_n\| = \|x - c\| \le d(x, C) + \varepsilon,$$

and so

$$\limsup d(x, C_n) \le d(x, C).$$

Suppose now that $\liminf d(x, C_n) < d(x, C) - \varepsilon$. Select $c_{n'} \in C_{n'}$ with $\|x - c_{n'}\| < d(x, C) - \varepsilon$. Since X is reflexive $\{c_{n'}\}$ has a weak subsequential limit point x_0 which, by M(ii), lies in C. Thus, as the norm is weakly lower semicontinuous

$$d(x, C) \leq \|x - x_0\| \leq d(x, C) - \varepsilon,$$

a contradiction.

(b) Suppose X is not reflexive. Select e in X with $\|e\| = 1$. Consider $K := B[0, 1] \cap B[e, \frac{1}{2}]$. (Here $B[x, r]$ is the closed ball of radius r around x.) Since K contains $B[\frac{3}{4}e, \frac{1}{4}]$, K is not countably weakly compact and we can find e_n with $\|e_n\| \leq 1$, $\|e - e_n\| \leq \frac{1}{2}$ so that $\{e_n\}$ has no weak cluster point [Da]. We now let $C_n := ordinary\{0, e_n\}$ and $C := \{0\}$.

We first establish that $\lim_M C_n = C$. Since $0 \in C_n$ for all n, it suffices to show that if $0 \leq t_{n'} \leq 1$ and $t_{n'}e_{n'}$ converges weakly to x then $x = 0$. We may assume, on extracting a further subsequence, that $t_{n'}$ converges to t. If $t = 0$ then $x = 0$ since $\|e_n\| \leq 1$. If $t > 0$ then $e_{n'}$ converges weakly to $t^{-1}x$, which is impossible.

Suppose that $\lim_W C_n = C'$. For any $y \in C'$ we have $d(y, C_n) \to 0$. By M(ii) we see that $y = 0$. Thus $C' = C = \{0\}$. However,

$$d(e, C_n) \leq \|e_n - e\| \leq \tfrac{1}{2}$$

while $d(e, C) = \|e\| = 1$ and $\lim_W C_n \neq C$. □

Part (a) of the previous result is well known (see [So], p. II.6).

Corollary 2.3. *A Banach space X is reflexive if and only if Mosco convergence implies Wijsman convergence for sequences of closed convex subsets of X.*

This corollary explains the need for reflexivity in our main results. The next result similarly explains the need for convexity of the subsets.

Theorem 2.4. *A Banach space X is finite dimensional if and only if for every sequence $\{C_n\}$ of weakly closed subsets of X, and weakly closed set C*

$$\lim_W C_n = C \quad \textit{if and only if} \quad \lim_M C_n = C.$$

Proof. First, suppose X is finite dimensional. Then Wijsman and Mosco convergences coincide for closed sets [Au, p. 244]. Second, Theorem 2.2(b) covers the nonreflexive case.

Suppose now that X is reflexive and infinite dimensional. Since 0 is in the weak closure of the unit sphere, we can choose a sequence $\{e_n\}$ of unit vectors converging weakly to zero (by the Kaplansky-Whitley construction, [Da, p. 58]). Select a norm-one linear functional f, and define

$$C := \{x : |f(x)| \geq \tfrac{1}{2}\} \quad \text{and} \quad C_n := C \cup \{e_n\}.$$

An easy computation shows that

$$d(x, C) = \max\{0, \tfrac{1}{2} - |f(x)|\}$$

and

$$d(x, C_n) = \min\{d(x, C), \|x - e_n\|\} \leq d(x, C).$$

Observe that $\liminf \|x - e_n\| \geq \max\{\|x\|,\ 1 - \|x\|\} \geq \frac{1}{2} \geq d(x, C)$ for all x (because $\{e_n\}$ converges weakly to zero and the norm is weakly lower semicontinuous). This shows that $\liminf d(x, C_n) \geq d(x, C)$ and so $\lim_W C_n = C$. Also C and each C_n is weakly closed. However, $e_n \in C_n$ and $\{e_n\}$ converges weakly to $0 \notin C$. Thus $\lim_M C_n \neq C$. □

Remark 2.5. A sequence $\{C_n\}$ of subsets of X converges Mosco [Wijsman] to a set C if and only if $\{\operatorname{cl} C_n\}$ converges Mosco [Wijsman] to C. Thus there is no loss of generality in considering only norm-closed sets.

3. Dual Kadec spaces

Three definitions are needed. First recall that the *duality map* J between X and X^* is the subgradient of $\frac{1}{2}\|\ \|^2$; explicitly $x^* \in J(x)$ if and only if $\|x^*\|^2 = \|x\|^2 = \langle x^*, x\rangle$. Also J is said to *norm-usco* provided that J is norm to norm upper semicontinuous with norm compact images (see [GGS]). In particular, if either the norm is Fréchet or X is finite dimensional then J is norm-usco. A dual Banach space X^* is (sequentially) *weak* Kadec* if whenever $\{x_n^*\}$ is a sequence of norm-one elements of X^* converging weak* to a norm-one element x^* then $\lim \|x_n^* - x^*\| = 0$.

Theorem 3.1. *Let X be a reflexive Banach space. The following statements are equivalent:*

(1) *X^* is (sequentially) weak* Kadec.*
(2) *The duality map on X is norm-usco.*
(3) *If $x \in X/\{0\}$ and $x_n \in X$ converge weakly to x then there is θ in $]0, 1]$ such that $\limsup \|x - \theta x_n\| < \|x\|$.*
(4) *If $x \in X/\{0\}$ and $x_n \in X$ converge weakly to x then there is θ in $]0, 1]$ such that $\liminf \|x - \theta x_n\| < \|x\|$.*
(5) *If $\{C_n\}$ is a sequence of closed convex subsets of X, for which $\lim_W C_n$ exists and is closed, then $\lim_M C_n = \lim_W C_n$.*
(6) *If f_n and f are elements of X^* and $\lim_W f_n^{-1}(1) = f^{-1}(1)$, then $\lim_M f_n^{-1}(1) = f^{-1}(1)$.*
(7) *If $x_n \in X$ converge weakly to x and $f_n \in X^*$ converge weak* to f such that $\|f_n\| = \|f\| = \langle f_n, x_n\rangle = 1$, then $\langle f, x\rangle = 1$.*

Proof. (1) ⇒ (2). Let $\{x_n\}$ converge in norm to x and $f_n \in J(x_n)$. Then, as J is locally bounded, there is a subsequence $\{f_{n'}\}$ weakly converging to some f. Since

$$\|f_{n'}\|^2 = \|x_{n'}\|^2 = \langle f_{n'}, x_n\rangle$$

while $x_{n'} \to x$ we have

$$\langle f, x\rangle = \lim\langle f_{n'}, x\rangle = \lim\langle f_{n'}, x_{n'}\rangle = \lim \|x_{n'}\|^2 = \|x\|^2 .$$

So $\|f\| \geq \lim \|f_{n'}\|$ and so, by weak lower semicontinuity of the dual norm, $\|f\| = \lim \|f_{n'}\|$. By the Kadec property, $\|f_{n'} - f\| \to 0$.

Some consideration shows that this implies (a) that for all $\varepsilon > 0$ there is $\delta > 0$ with $J(x) + B[0,\varepsilon] \supseteq J(B[x,\delta])$; and (b) that $J(x)$ is norm-compact.

(2) $\Rightarrow$ (3). Let $x \neq 0$ and $\{x_n\}$ converge weakly to x. Let $\varepsilon > 0$ with $\varepsilon \sup \|x_n\| < \|x\|^2$. Pick θ in $]0,1]$ and $f_n \in J(x - \theta x_n)$ so that $d(f_n, J(x)) < \varepsilon$ for all n, as is possible since $\{x_n\}$ is bounded and J is norm-usco. Choose a subsequence $\{x_{n'}\}$ with $\lim \|x - \theta x_{n'}\| = \limsup \|x - \theta x_n\|$. Select $g_n \in J(x)$ with $\|f_n - g_n\| < \varepsilon$ for all n. Since $J(x)$ is compact we may assume that $\{g_{n'}\}$ converges in norm to some $g \in J(x)$. Eventually we have $\|f_{n'} - g\| < \varepsilon$.

We have, from the subgradient property of J,

$$\begin{aligned}\{\|x - \theta x_{n'}\|^2 - \|x\|^2\}/2 &\leq -\langle f_{n'}, \theta x_{n'}\rangle \\ &= -\langle f_{n'} - g, \theta x_{n'}\rangle - \langle g, \theta x_{n'}\rangle \\ &\leq -\theta\langle g, x_{n'}\rangle + \theta\varepsilon\|x_{n'}\|.\end{aligned}$$

Thence

$$\lim\{\|x - \theta x_{n'}\|^2 - \|x\|^2\}/2 \leq \theta(\varepsilon \sup \|x_{n'}\| - \|x\|^2) < 0,$$

which shows that $\|x\| > \lim \|x - \theta x_{n'}\| = \limsup \|x - \theta x_n\|$.

(3) $\Rightarrow$ (4). This is immediate.

(4) $\Rightarrow$ (5). Let C_n and C be closed convex sets with $d(x, C_n) \to d(x, C)$ for all x. If $C = \varnothing$ then Proposition 2.1(a) shows $\lim_M C_n = \varnothing$. If $x \in C$ then $d(x, C) = 0$ so that $d(x, C_n) \to 0$ and one can find $c_n \in C_n$ with $c_n \to x$. Hence M(i) holds. Now suppose that $x_{n'} \in C_{n'}$ and $x_{n'}$ converges weakly to x. We must show that $x \in C$. Otherwise we let $c \neq x$ be the nearest point in C to x (C is convex and closed, and so proximinal). By translation we arrange that $c = 0$, so that $d(x, C) = \|x\|$ and $0 \in C$. As above we can find $c_n \in C_n$ with $c_n \to 0$. Let $y_{n'} := x_{n'} - c_{n'}$ which converges weakly to $x \neq 0$.

We apply (4) to $\{y_{n'}\}$ and x. For θ as promised we observe that

$$\theta y_{n'} + c_{n'} = \theta(x_{n'} - c_{n'}) + c_{n'} \in C_{n'}$$

and so

$$d(x, C_{n'}) - \|x\| \leq \|x - (\theta y_{n'} + c_{n'})\| - \|x\| \leq \|c_{n'}\| + \|x - \theta y_{n'}\| - \|x\|.$$

Thus

$$\begin{aligned}0 = d(x, C) - \|x\| = \lim d(x, C_{n'}) - \|x\| &\leq \liminf[\|c_{n'}\| + \|x - \theta y_{n'}\| - \|x\|] \\ &= \liminf[\|x - \theta y_{n'}\| - \|x\|] < 0,\end{aligned}$$

a clear contradiction. Thus $x \in C$, giving M(ii).

(5) $\Rightarrow$ (6). Let $C_n := f_n^{-1}(1)$ and $C := f^{-1}(1)$.

(6) $\Rightarrow$ (7). We have $x_n \in X$ converging weakly to x and $f_n \in X^*$ converging weak* to f such that $\|f_n\| = \|f\| = \langle f_n, x_n\rangle = 1$. Let $C_n := f_n^{-1}(1)$ and $C := f^{-1}(1)$. A computation shows that

$$d(x, C_n) = |1 - \langle f_n, x\rangle| \quad \text{and} \quad d(x, C) = |1 - \langle f, x\rangle|,$$

so that $d(x, C_n) \to d(x, C)$ for all x. Thus by (6) $\lim_M C_n = C$. Now $x_n \in C_n$ and $\{x_n\}$ converges weakly to x, so that $x \in C$ and $\langle f, x\rangle = 1$.

(7) $\Rightarrow$ (1). We will establish (1) in the form:

(1′) If $\|f_n\| = \|f\| = 1$ and $\{f_n\}$ converges weak* to f, then $\{f_n\}$ converges to f uniformly on weakly compact sets. Since X is reflexive, (1′) is equivalent to (1).

Suppose (1′) fails. Then select a weakly compact set W and a subsequence $\{f_{n'}\}$ of $\{f_n\}$ with

$$\sup_{w \in W} |\langle f_{n'} - f, w\rangle| > \varepsilon > 0.$$

Let $x_{n'} \in W$ with $|\langle f_{n'} - f, x_{n'}\rangle| > \varepsilon$, and extracting subsequences we may assume that $\{x_{n'}\}$ converges weakly to x and $\langle f_{n'}, x_{n'}\rangle \to \alpha \neq \langle f, x\rangle$. Then

$$y_{n'} := \frac{x_{n'} - x}{\langle f_{n'}, x_{n'} - x\rangle} \quad \text{converges weakly to } 0,$$

and since $\langle f_{n'}, y_{n'}\rangle = 1 = \|f_{n'}\| = \|f\|$ and $\{y_{n'}\}$ converges weakly to 0, we obtain from (7) that $\langle f, 0\rangle = 1$. This contradiction establishes (1′) and completes our circuit. □

Remark 3.2. (a) Let us reiterate that these equivalences hold for any finite-dimensional normed space [obvious from (3)] or when the norm is Fréchet differentiable [obvious from (2)].

(b) Note in a reflexive space that there is a complete duality between the Kadec condition (1) and the Tsukada conditions (3) and (4).

(c) In each of conditions (3) and (4) " $<$ " may be replaced by " $\leq$ ", as follows if X_n is replaced by $2x_n - x$ in the nonstrict version.

Corollary 3.3. *If X is a smooth reflexive normed space then the norm on X is Fréchet if and only if the equivalences* (1) *through* (7) *of Theorem* 3.1 *hold.*

Proof. Since X is reflexive, the norm is Fréchet precisely when the dual norm is strictly convex and Kadec, or equivalently when the original norm is smooth and the dual norm is Kadec [Ho, Ts, Bo-Fi]. □

Theorem 3.4. *A Banach space is reflexive and dual Kadec if and only if Mosco and Wijsman convergences coincide for sequences of closed convex sets.*

Proof. Combine the equivalence of (1) and (5) of Theorem 3.1 with Corollary 2.3. □

The next result extends and unifies Theorem 3.3 of Tsukada [Ts].

Corollary 3.5. *If the norm on X^* is Fréchet differentiable* (*so X is reflexive*) *and Kadec, then the following are equivalent for any sequence $\{C_n\}$ of closed nonempty convex subsets of X.*

(1) $\lim_M C_n$ *exists and is nonempty.*
(2) $\lim_W C_n$ *exists and is nonempty.*
(3) *For every $x \in X$ the sequence of metric projections of x onto C_n is norm convergent.*

Proof. This follows by Tsukada's arguments [Ts, pp. 306–307] on replacing his Theorem 2.5 by our Theorem 3.4. □

We note that one can explicitly give norms on l_2 which are smooth but for which the dual is not Kadec: let $\|\ \|$ be the Hilbert norm on l_2 and set $|||(r,x)|||^2 := \|(r,Tx)\|^2 + \max\{\|x\|,|r|\}^2$ where $(Tx)_n := x_n/n$. Then the dual norm is smooth but its dual is not Kadec. So we have a smooth equivalent norm on Hilbert space for which Mosco and Wijsman convergences do not coincide for sequences of closed convex sets.

We finish this section by noting a remarkable duality.

Theorem 3.6. *Mosco and Wijsman convergences coincide for sequences of closed convex sets in a Banach space* X *if and only if every closed nonempty subset* C *of* X^* *is almost proximinal (i.e. there is a generic set of points in* $X^*\backslash C$ *which admit nearest points in* C).

Proof. By the theorem of Lau [La] and Konjagin [Ko], also derived in [Bo-Fi, Theorem 5.11], the second condition also coincides, with X^* being reflexive and Kadec. □

4. The nonreflexive case

While the results of §2 show that many things fail in the absence of reflexivity, there is an adequate analogue for Theorem 3.1.

Theorem 4.1. *Let* X *be a Banach space whose dual unit ball is weak** *sequentially compact. The following statements are equivalent.*

(1) *If* f_n *and* f *are elements of* X^* *with* $\|f_n\| = \|f\| = 1$ *and if* $\{f_n\}$ *converges weak** *to* f *while* f *is norm-attaining, then* $\{f_n\}$ *converges to* f *uniformly on weakly compact sets, that is to say in the Mackey topology* $\tau := \tau(X^*, X)$.

(2) *The duality map on* X *is norm-*τ *upper semicontinuous with sequentially* τ*-compact images.*

(3) *If* $x \in X/\{0\}$ *and* $x_n \in X$ *converge weakly to* x *then there is* θ *in* $]0,1]$ *such that* $\limsup \|x - \theta x_n\| < \|x\|$.

(4) *If* $x \in X/\{0\}$ *and* $x_n \in X$ *converge weakly to* x *then there is* θ *in* $]0,1]$ *such that* $\liminf \|x - \theta x_n\| < \|x\|$.

(5) *If* $\{C_n\}$ *is a sequence of closed convex subsets of* X, *for which* $\lim_W C_n$ *exists and is proximinal, then* $\lim_M C_n = \lim_W C_n$.

(6) *If* f_n *and* f *are elements of* X^* *with* f *norm-attaining and* $\lim_W f_n^{-1}(1) = f^{-1}(1)$, *then* $\lim_M f_n^{-1}(1) = f^{-1}(1)$.

(7) *If* $x_n \in X$ *converge weakly to* x *and* $f_n \in X^*$ *converge weak** *to a norm-attaining element* $f \in X^*$ *such that* $\|f_n\| = \|f\| = \langle f_n, x_n\rangle = 1$, *then* $\langle f, x\rangle = 1$.

Proof. The proof proceeds essentially as in Theorem 3.1, replacing the norm topology on X^* by the Mackey topology τ (see [GGS]). □

For any Banach space with an equivalent smooth norm the dual unit ball is weak* sequentially compact. This holds whenever the space is WCG and so if X is separable or reflexive.

Corollary 4.2. *Let X be a Banach space whose dual unit ball is weak* sequentially compact. Supposes that either* (1) *the dual norm is sequentially weak* Kadec or* (2) *the norm is Fréchet differentiable. If $\{C_n\}$ is a sequence of closed convex subsets of X, and $\lim_W C_n$ exists and is proximinal, then $\lim_M C_n = \lim_W C_n$.*

Proof. (1) and (2) are stronger than (1) and (2) of Theorem 4.1. □

Example 4.3. (a) Let X be $c_0(S)$ for any set S, endowed with the supremum norm. Then Corollary 4.2(1) holds.

(b) Let X be $c_0(S)$, endowed with any Fréchet differentiable renorm. Then Corollary 4.2(2) holds.

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