REMARKS ON THE RIGIDITY AND STABILITY OF MINIMAL SUBMANIFOLDS

MAKOTO SAKAKI

(Communicated by Jonathan M. Rosenberg)

ABSTRACT. We improve the pinching theorem of Simons and the stability theorem of Barbosa and do Carmo with an elementary method.

Simons [7] proved a pinching theorem for closed minimal submanifolds in the unit sphere, which led to an intrinsic rigidity result. In this note, using an elementary method, we improve his theorem and obtain a result which does not depend on the dimension of the ambient space.

Lemma. Let $M$ be an $m$-dimensional minimal submanifold in a space of constant curvature $a$. Let $A$ and $\Delta$ denote the second fundamental form and the Laplacian of $M$, respectively. Then $-\langle A, \Delta A \rangle \leq \langle 2 - 2/(m - 1)(m + 2) \rangle |A|^4 - ma|A|^2$.

Proof. We use the argument of Chern, do Carmo and Kobayashi [3]. We assume that the ambient space is $n$-dimensional. Set $q = 2^{-1} m(m+1) - 1 = 2^{-1} (m - 1)(m + 2)$. When $n \leq m + q$, the Lemma is included in [3]. So we assume that $n > m + q$ in the following. We make a pointwise argument at a point $p$ on $M$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for the tangent space of the ambient space at $p$ such that $e_1, \ldots, e_m$ are tangent to $M$. We make use of the following convention on the ranges of indices: $1 \leq i, j \leq m$, $m + 1 \leq \alpha, \beta \leq n$, $m + 1 \leq \xi, \eta \leq m + q$. Let $h_{ij}^\alpha$ be the components of $A$ with respect to the basis. Set $T_{\alpha\beta} = \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta$ and $T_\alpha = T_{\alpha\alpha}$.

It is an elementary observation that at each point the dimension of the image of the second fundamental form of an $m$-dimensional minimal submanifold is at most $2^{-1} m(m+1) - 1 = q$. Thus we may choose $e_{m+1}, \ldots, e_n$ so that $h_{ij}^\alpha = 0$ for $\alpha > m + q$. Let $V$ be a subspace of the normal space of $M$ at $p$ spanned by $e_{m+1}, \ldots, e_{m+q}$. We define a symmetric linear transformation $T$ of $V$ by $T(\sum_\eta v^\eta e_\eta) = \sum_\xi T_\xi v^\eta e_\xi$, which is well defined. As $T$ is symmetric, we may change $e_{m+1}, \ldots, e_{m+q}$ so that the $(q \times q)$-matrix $(T_{\xi\eta})$ is diagonal. Then

Received by the editors July 11, 1988 and, in revised form, November 17, 1988.

1980 Mathematics Subject Classification (1985 Revision). Primary 53C40; Secondary 53C20, 53A10.
apparently the \((n - m) \times (n - m)\)-matrix \((T_{\alpha\beta})\) is diagonal. So we can see that the equation (3.5) and the inequality (3.7) of [3] are valid with respect to this basis. Therefore, we obtain

\[
-\langle A, \Delta A \rangle \leq 2 \sum_{\xi \neq \eta} T_{\xi} T_{\eta} + \sum_{\xi} T_{\xi}^2 - ma |A|^2 = 2 \left( \sum_{\xi} T_{\xi} \right)^2 - \sum_{\xi} T_{\xi}^2 - ma |A|^2 \\
\leq 2 \left( \sum_{\xi} T_{\xi} \right)^2 - \frac{1}{q} \left( \sum_{\xi} T_{\xi} \right)^2 - ma |A|^2 = \left( 2 - \frac{1}{q} \right) |A|^4 - ma |A|^2
\]

(cf. the proof of Lemma 5.3.1 of [7]). Thus the proof is complete.

Referring to [7] and [3] with the Lemma, we obtain the following:

**Theorem 1.** Let \(M\) be an \(m\)-dimensional closed minimal submanifold in the unit sphere. Suppose that the scalar curvature \(S\) of \(M\) satisfies \(m(m-1)(2m^2 + m - 8)/2(m^2 + m - 3) \leq S \leq m(m-1)\). Then either (i) \(S = m(m-1)\) and \(M\) is totally geodesic, or (ii) \(S = \frac{2}{3}\) and \(M\) is the Veronese surface in a totally geodesic 4-sphere.

By use of the same argument as above, we can improve the results of Yau [8] and Pan [5]. For example:

**Theorem 2.** Let \(M\) be an \(m\)-dimensional complete submanifold with parallel mean curvature in the unit sphere. Suppose that the second fundamental form \(A\) of \(M\) satisfies \(3 + \sqrt{m-2}/(m-1)(m+2)\)||\(A|^2 \leq m - \varepsilon\) for a positive constant \(\varepsilon\). Then \(M\) lies in a totally geodesic \((m + 1)\)-sphere.

Using the inequality of the Lemma for \(m = 2\), we can improve the stability theorem of Barbosa and do Carmo [2].

**Theorem 3.** Let \(M\) be a minimal surface in the simply-connected space form of constant curvature \(a\), and let \(D\) be a simply-connected compact domain with piecewise smooth boundary on \(M\). Let \(A\) denote the second fundamental form of \(M\). If \(\int_D (|A| + |A|^2/2) \, dM < 4\pi/3\), then \(D\) is stable.

**Remark.** (i) When \(a \geq 0\), Theorem 3 is proved in a little different way (cf. [1], Hoffman and Osserman [4]).

(ii) The proof of Lemma 2.11 of [2] is incorrect. In the proof of the lemma, we may choose the basis so that the components \(h_{ij}^\alpha\) satisfy

\[
(h_{ij}^3) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad (h_{ij}^4) = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, \quad (h_{ij}^5) = \cdots = (h_{ij}^n) = 0
\]

for some \(\lambda\) and \(\mu\) (cf. the proof of the Lemma above). The lemma is shown with the help of this fact.

(iii) In a succeeding paper [6] we will generalize Theorem 3 for a general ambient space.

**Acknowledgment**

The author wishes to thank the referee for useful suggestions.
RIGIDITY AND STABILITY OF MINIMAL SUBMANIFOLDS

References


Department of Mathematics, Tokyo Institute of Technology, Oh-Okayama, Meguro-Ku, Tokyo 152, Japan