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REMARKS ON THE RIGIDITY AND STABILITY OF MINIMAL SUBMANIFOLDS

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ABSTRACT. We improve the pinching theorem of Simons and the stability theorem of Barbosa and do Carmo with an elementary method.

Simons [7] proved a pinching theorem for closed minimal submanifolds in the unit sphere, which led to an intrinsic rigidity result. In this note, using an elementary method, we improve his theorem and obtain a result which does not depend on the dimension of the ambient space.

Lemma. *Let M be an m-dimensional minimal submanifold in a space of constant curvature a. Let A and Δ denote the second fundamental form and the Laplacian of M, respectively. Then $-\langle A, \Delta A\rangle \leq \{2 - 2/(m-1)(m+2)\}|A|^4 - ma|A|^2$.*

Proof. We use the argument of Chern, do Carmo and Kobayashi [3]. We assume that the ambient space is n-dimensional. Set $q = 2^{-1}m(m+1) - 1 = 2^{-1}(m-1)(m+2)$. When $n \leq m+q$, the Lemma is included in [3]. So we assume that $n > m+q$ in the following. We make a pointwise argument at a point p on M. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for the tangent space of the ambient space at p such that $e_1, \ldots, e_m$ are tangent to M. We shall make use of the following convention on the ranges of indices: $1 \leq i$, $j \leq m$, $m+1 \leq \alpha$, $\beta \leq n$, $m+1 \leq \xi$, $\eta \leq m+q$. Let h_{ij}^{α} be the components of A with respect to the basis. Set $T_{\alpha\beta} = \sum_{i,j} h_{ij}^{\alpha} h_{ij}^{\beta}$ and $T_{\alpha} = T_{\alpha\alpha}$.

It is an elementary observation that at each point the dimension of the image of the second fundamental form of an m-dimensional minimal submanifold is at most $2^{-1}m(m+1)-1 = q$. Thus we may choose $e_{m+1}, \ldots, e_n$ so that $h_{ij}^{\alpha} = 0$ for $\alpha > m+q$. Let V be a subspace of the normal space of M at p spanned by $e_{m+1}, \ldots, e_{m+q}$. We define a symmetric linear transformation T of V by $T(\sum_{\eta} v^{\eta} e_{\eta}) = \sum_{\xi,\eta} T_{\xi\eta} v^{\eta} e_{\xi}$, which is well defined. As T is symmetric, we may change $e_{m+1}, \ldots, e_{m+q}$ so that the $(q \times q)$-matrix $(T_{\xi\eta})$ is diagonal. Then

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apparently the $(n-m)\times(n-m)$-matrix $(T_{\alpha\beta})$ is diagonal. So we can see that the equation (3.5) and the inequality (3.7) of [3] are valid with respect to this basis. Therefore, we obtain

$$\begin{aligned}-\langle A, \Delta A\rangle &\leq 2\sum_{\xi\neq\eta} T_\xi T_\eta + \sum_\xi T_\xi^2 - ma|A|^2 = 2\left(\sum_\xi T_\xi\right)^2 - \sum_\xi T_\xi^2 - ma|A|^2\\ &\leq 2\left(\sum_\xi T_\xi\right)^2 - \frac{1}{q}\left(\sum_\xi T_\xi\right)^2 - ma|A|^2 = \left(2-\frac{1}{q}\right)|A|^4 - ma|A|^2\end{aligned}$$

(cf. the proof of Lemma 5.3.1 of [7]). Thus the proof is complete.

Referring to [7] and [3] with the Lemma, we obtain the following:

Theorem 1. *Let M be an m-dimensional closed minimal submanifold in the unit sphere. Suppose that the scalar curvature S of M satisfies $m(m-1)(2m^2+m-8)/2(m^2+m-3)\leq S\leq m(m-1)$. Then either* (i) $S=m(m-1)$ *and M is totally geodesic, or* (ii) $S=\frac{2}{3}$ *and M is the Veronese surface in a totally geodesic 4-sphere.*

By use of the same argument as above, we can improve the results of Yau [8] and Pan [5]. For example:

Theorem 2. *Let M be an m-dimensional complete submanifold with parallel mean curvature in the unit sphere. Suppose that the second fundamental form A of M satisfies $\{3+\sqrt{m}-2/(m-1)(m+2)\}|A|^2\leq m-\varepsilon$ for a positive constant ε. Then M lies in a totally geodesic $(m+1)$-sphere.*

Using the inequality of the Lemma for $m=2$, we can improve the stability theorem of Barbosa and do Carmo [2].

Theorem 3. *Let M be a minimal surface in the simply-connected space form of constant curvature a, and let D be a simply-connected compact domain with piecewise smooth boundary on M. Let A denote the second fundamental form of M. If $\int_D(|a|+|A|^2/2)\,dM<4\pi/3$, then D is stable.*

Remark. (i) When $a\geq 0$, Theorem 3 is proved in a little different way (cf. [1], Hoffman and Osserman [4]).

(ii) The proof of Lemma 2.11 of [2] is incorrect. In the proof of the lemma, we may choose the basis so that the components h_{ij}^α satisfy

$$(h_{ij}^3)=\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix},\qquad (h_{ij}^4)=\begin{pmatrix}0 & \mu\\ \mu & 0\end{pmatrix},\qquad (h_{ij}^5)=\cdots=(h_{ij}^n)=0$$

for some λ and μ (cf. the proof of the Lemma above). The lemma is shown with the help of this fact.

(iii) In a succeeding paper [6] we will generalize Theorem 3 for a general ambient space.

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