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ON TWO-GENERATOR DISCRETE GROUPS OF MÖBIUS TRANSFORMATIONS

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ABSTRACT. Assume that Möbius transformations f and g generate a discrete group. We obtain the following generalizations of Jørgensen's inequalities. If $\operatorname{tr}(fgf^{-1}g^{-1}) \neq 1$, then $|\operatorname{tr}^2(f) - 2| + |\operatorname{tr}(fgf^{-1}g^{-1}) - 1| \geq 1$. If $\operatorname{tr}(fgf^{-1}g^{-1}) = 1$, then either $\operatorname{tr}^2(f) = 2$ or $\operatorname{tr}^2(f) = 1$ or $|\operatorname{tr}^2(f) - 2| > \frac{1}{2}$ and $|\operatorname{tr}^2(f) - 1| > \frac{1}{2}$. If $\operatorname{tr}^2(f) \neq 1$, then $|\operatorname{tr}^2(f) - 1| + |\operatorname{tr}(fgf^{-1}g^{-1})| \geq 1$. If $\operatorname{tr}^2(f) = 1$ then either $\operatorname{tr}(fgf^{-1}g^{-1}) = 0$ or $\operatorname{tr}(fgf^{-1}g^{-1}) = 1$ or $|\operatorname{tr}(fgf^{-1}g^{-1})| > \frac{1}{2}$ and $|\operatorname{tr}(fgf^{-1}g^{-1}) - 1| > \frac{1}{2}$.

1. INTRODUCTION

Let f and g be two Möbius transformations that generate a nonelementary discrete group. Then they satisfy Jørgensen's well-known inequality [5]:

$$|\operatorname{tr}^2(f) - 4| + |\operatorname{tr}(fgf^{-1}g^{-1}) - 2| \geq 1. \tag{1}$$

It is interesting to generalize Jørgensen's inequality. There are many papers concerning such generalizations, for example, those published by Brooks and Matelski [2], Gilman [4] and Rosenberger [9]. These results suggest the following question: Do there exist other constants a and b such that

$$|\operatorname{tr}^2(f) - a| + |\operatorname{tr}(fgf^{-1}g^{-1}) - b| \geq 1? \tag{2}$$

In this paper, by refining the Jørgensen's Lie product technique, we obtain several inequalities of this kind for certain discrete groups. In fact, we establish the following results:

Theorem 1. *Suppose that the Möbius transformations f and g generate a discrete group. If $\operatorname{tr}(fgf^{-1}g^{-1}) \neq 1$, then*

$$|\operatorname{tr}^2(f) - 2| + |\operatorname{tr}(fgf^{-1}g^{-1}) - 1| \geq 1; \tag{3}$$

if $\operatorname{tr}(fgf^{-1}g^{-1}) = 1$ and $\operatorname{tr}^2(f) \neq 2$, then

$$|\operatorname{tr}^2(f) - 2| > \tfrac{1}{2}. \tag{4}$$

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Theorem 2. *Suppose that the Möbius transformations f and g generate a discrete group. If* $\operatorname{tr}^2(f) \neq 1$, *then*

$$|\operatorname{tr}^2(f) - 1| + |\operatorname{tr}(fgf^{-1}g^{-1})| \geq 1; \tag{5}$$

if $\operatorname{tr}^2(f) = 1$, *then*

$$|\operatorname{tr}(fgf^{-1}g^{-1})| > \tfrac{1}{2} \quad \text{or} \quad \operatorname{tr}(fgf^{-1}g^{-1}) = 0; \tag{6}$$

and

$$|\operatorname{tr}(fgf^{-1}g^{-1}) - 1| > \tfrac{1}{2} \quad \text{or} \quad \operatorname{tr}(fgf^{-1}g^{-1}) = 1. \tag{7}$$

Theorem 3. *Suppose that the Möbius transformations f and g generate a discrete group. If* $\operatorname{tr}(fgf^{-1}g^{-1}) \neq 1$, *then*

$$|\operatorname{tr}^2(f) - \operatorname{tr}(fgf^{-1}g^{-1})| + |\operatorname{tr}(fgf^{-1}g^{-1}) - 1| \geq 1; \tag{8}$$

if $\operatorname{tr}(fgf^{-1}g^{-1}) = 1$ *and* $\operatorname{tr}^2(f) \neq 1$, *then*

$$|\operatorname{tr}^2(f) - 1| > \tfrac{1}{2}. \tag{9}$$

Remark 1. If we take

$$f(z) = iz, \qquad g(z) = 2z,$$

then the lower bounds of (3) and (8) are attained. We see also (5) is sharp by taking

$$f(z) = -z, \qquad g(z) = \frac{z-1}{z+1}. \tag{10}$$

The recent work of Maskit in response to a question of Gehring and Martin (see [3]) shows that inequalities (3), (5) and (8) are sharp for nonelementary groups [8]. However, the lower-bound halves of (4), (6), (7) and (9) are not the best possible. In fact, the above theorems are equivalent under the Lie product transformation.

Remark 2. Take the discrete group G_1 generated by

$$f(z) = iz, \qquad g(z) = \frac{z-1}{z+1}, \tag{11}$$

the discrete group G_2 generated by

$$f(z) = \frac{z-i}{z+i}, \qquad g(z) = -\frac{z-i}{z+i}, \tag{12}$$

and the discrete group G_3 generated by

$$f(z) = \frac{z-i}{z+i}, \qquad g(z) = \frac{iz+1}{z+i}. \tag{13}$$

The group G_i shows that hypothesis for Theorem i is necessary.

Remark 3. Gehring and Martin have worked on the same problem and have established a general result which contains many inequalities including (3) and (8) as well as Jørgensen's inequality [3].

2. Proof of the theorems

The proof is based on the following lemmas:

Lemma 1. *Suppose that the Möbius transformations f and g generate a discrete group. If $\operatorname{tr}(f) \neq 0$ and $\operatorname{tr}(fgfg^{-1}) \neq 2$, then*

$$|\operatorname{tr}^2(f)| + |\operatorname{tr}(fgf^{-1}g^{-1}) + 2| \geq 1. \tag{14}$$

Lemma 2. *Suppose that the Möbius transformations f and g generate a discrete group. If $\operatorname{tr}(fgfg^{-1}) \neq 1$, then*

$$|\operatorname{tr}^2(f) - 2| + |\operatorname{tr}(fgfg^{-1}) - 1| \geq 1; \tag{15}$$

if $\operatorname{tr}(fgfg^{-1}) = 1$ and $\operatorname{tr}^2(f) \neq 2$, then

$$|\operatorname{tr}^2(f) - 2| > \tfrac{1}{2}. \tag{16}$$

We will prove these lemmas in §3. In this section we use Lemma 2 to prove Theorems 1–3.

Let the matrices A and B represent f and g in $SL(2, C)$ respectively. If $\operatorname{tr}(ABA^{-1}B^{-1}) = 2$, then there is nothing to prove. Next we suppose that $\operatorname{tr}(ABA^{-1}B^{-1}) \neq 2$. Let $\phi = AB - BA$ be the Lie product of A and B, then ϕ is elliptic of order 2 and conjugates A and B to their inverses. The group $\langle A, B\rangle$ has index at most two in the group $\langle A, B, \phi\rangle$ and thus both groups are simultaneously discrete or nondiscrete. This is Jørgensen's Lie product construction [6 and 7].

Now applying Lemma 2 to A and $B\phi$ yields Theorem 1.

Notice that if we regard ϕ as a matrix, then $\phi^2 = -I$. If we take $F = AB\phi$ and $G = A^{-1}$, then

$$\begin{aligned} \operatorname{tr}^2(F) &= \operatorname{tr}(AB\phi)^2 + 2 = -\operatorname{tr}(ABA^{-1}B^{-1}) + 2, \\ \operatorname{tr}(FGF^{-1}G^{-1}) &= \operatorname{tr}(ABAB^{-1}), \\ \operatorname{tr}(FGFG^{-1}) &= -\operatorname{tr}(A^2) = -\operatorname{tr}^2(A) + 2. \end{aligned} \tag{17}$$

So applying Lemma 2 to $AB\phi$ and A^{-1} yields (5) and (6), while applying (6) to A and $B\phi$ yields (7). Then applying Theorem 1 to $AB\phi$ and A^{-1}, we have: if $\operatorname{tr}(ABA^{-1}B^{-1}) \neq 1$, then

$$|\operatorname{tr}(ABA^{-1}B^{-1})| + |\operatorname{tr}(ABAB^{-1}) - 1| \geq 1; \tag{18}$$

if $\operatorname{tr}(ABAB^{-1}) = 1$ and $\operatorname{tr}(ABA^{-1}B^{-1}) \neq 0$, then

$$|\operatorname{tr}(ABA^{-1}B^{-1})| > \tfrac{1}{2}. \tag{19}$$

Replacing A and B by A and $B\phi$ in (18) and (19), we get Theorem 3.

If we continue the same trick, we come back to (3), (5), or (15).

3. Proof of the lemmas

Simply applying the Lie product extension with $AB\phi$ and A^{-1} to Jørgensen's inequality in [7] yields Lemma 1.

For the proof of Lemma 2, we need only to consider the case where f is elliptic or loxodromic. Let f and g be represented respectively by the matrices A and B in $SL(2,C)$. The inequalities are conjugacy invariant and so we can assume A is of the form

$$A = \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix} \qquad (u \neq -1, 0, 1)$$

and

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (ad - bc = 1).$$

Then

$$\operatorname{tr}^2(A) = (u + 1/u)^2, \ \operatorname{tr}(ABAB^{-1}) = ad(u - 1/u)^2 + 2$$

and

$$\operatorname{tr}(ABA^{-1}B^{-1}) = u^2 + 1/u^2 - ad(u - 1/u)^2. \tag{20}$$

Now we suppose that $\operatorname{tr}(ABAB^{-1}) \neq 1$ and (15) is false. Then

$$\begin{aligned} r &= |\operatorname{tr}^2(A) - 2| + |\operatorname{tr}(ABAB^{-1}) - 1| \\ &= |u^2 + 1/u^2| + |1 + ad(u - 1/u)^2| < 1. \end{aligned} \tag{21}$$

Let

$$B_1 = B,$$

$$B_{n+1} = A_n B_n A_n^{-1} B_n^{-1} = \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} \qquad (n = 1, 2, 3, \dots).$$

Then we have

$$\begin{aligned} a_{n+1} &= a_n d_n (1 - u^2) + u^2, \\ b_{n+1} &= a_n b_n (u^2 - 1), \\ c_{n+1} &= c_n d_n \left(\frac{1}{u^2} - 1 \right), \\ d_{n+1} &= a_n d_n (1 - 1/u^2) + 1/u^2. \end{aligned} \tag{22}$$

Hence

$$1 + a_{n+1} d_{n+1} (u - 1/u)^2 = [1 + a_n d_n (u - 1/u)^2][u^2 + 1/u^2 - 1 - a_n d_n (u - 1/u)^2]. \tag{23}$$

Let $t = u^2 + 1/u^2$ and $z_n = 1 + a_n d_n (u - 1/u)^2$; then (23) becomes

$$z_{n+1} = z_n (t - z_n). \tag{24}$$

So

$$|z_2| \leq |z_1|(|t| + |z_1|) = |z_1| r. \tag{25}$$

Then by induction, we have

$$|z_{n+1}| \leq |z_n| r \leq |z_1| r^n. \tag{26}$$

Observe that z_n tends to zero as n tends to ∞. So from (22)

$$a_{n+1} = \frac{u^2(u^2 - z_n)}{u^2 - 1} \to \frac{u^4}{u^2 - 1}(n \to \infty), \tag{27}$$

$$d_{n+1} = \frac{(u^2 z_n - 1)}{u^2(u^2 - 1)} \to \frac{-1}{u^2(u^2 - 1)}(n \to \infty). \tag{28}$$

Now we consider two cases.

Case 1. Suppose that $z_n \neq 0$ for all n. Then (26) implies that the z_n are distinct and hence that the same is true of a_n and d_n.

If f is elliptic, then $|u| = 1$. Therefore for some constant K,

$$|b_{n+2}| = |u^2 - z_n|\,|b_{n+1}| \leq (1 + Kr^n)|b_{n+1}|$$

and

$$|c_{n+2}| = |1 - u^2 z_n|\,|c_{n+1}| \leq (1 + Kr^n)|c_{n+1}|. \tag{29}$$

Hence the sequences $\{b_n\}$, $\{c_n\}$ are bounded.

If f is loxodromic, then $|u| \neq 1$. Notice that the distinctness of $\{z_n\}$ implies for any n, $b_n \neq 0$. So we can choose an integer $k = k(n)$ such that

$$1 \leq |u^{2k} b_n| \leq |u^2| + |1/u^2|. \tag{30}$$

Thus

$$A^k B_n A^{-k} = \begin{pmatrix} a_n & u^{2k} b_n \\ u^{-2k} c_n & d_n \end{pmatrix}$$

has bounded entries.

Hence we can find a subsequence of $\langle A, B\rangle$ which is distinct and bounded. This contradicts discreteness.

The above argument explicitly follows Jørgensen. (Also see [1].)

Case 2. Suppose that $z_n = 0$ for some n.

The hypothesis $\mathrm{tr}(fgfg^{-1}) \neq 1$ means $z_1 \neq 0$. From (24) we can find an integer N such that

$$z_n = 0 \quad \text{for } n > N \quad \text{and } z_n \neq 0 \quad \text{for } n \leq N. \tag{31}$$

So

$$t = z_N \neq 0 \quad \text{and } |t| = \tfrac{1}{2}(|t| + |z_N|) \leq \tfrac{1}{2}(|t| + |z_1|) < \tfrac{1}{2}. \tag{32}$$

Also

$$a_n = \frac{u^4}{u^2 - 1} \quad \text{and } d_n = \frac{-1}{u^2(u^2 - 1)} \quad \text{for } n > N + 1.$$

Thus we get a matrix M in $\langle A, B\rangle$ of the form

$$M = \begin{pmatrix} \frac{u^4}{u^2-1} & * \\ * & \frac{-1}{u^2(u^2-1)} \end{pmatrix}. \tag{33}$$

By computation

$$M^2 = \begin{pmatrix} \frac{u^6+u^4+1}{u^2-1} & * \\ * & -\frac{u^6+u^2+1}{u^4(u^2-1)} \end{pmatrix}. \tag{34}$$

Now let B_1 be replaced by M^2 and return to the iteration of B_n. Let z_n be defined as above corresponding to the new sequence. Then

$$\begin{aligned} z_1 &= 1 + a_1 d_1 (u - 1/u)^2 \\ &= 1 - \frac{(u^6+u^4+1)(u^6+u^2+1)}{u^6} \\ &= -t(t^2+t-2) \end{aligned} \tag{35}$$

and

$$z_2 = z_1(t - z_1) = -t^2(t-1)(t+2)(t^2+t-1). \tag{36}$$

Let

$$D_1 = \{t\colon 0 < |t| \le \tfrac{1}{2} \quad \text{and} \quad |t(t-1)(t+2)(t^2+t-1)| < 1\}. \tag{37}$$

If $t \in D_1$, we have $0 < |z_2| < |t|$. From (24)

$$|z_n|(|t| - |z_n|) \le |z_{n+1}| \le |z_n|(|t| + |z_n|).$$

By induction we get

$$0 < |z_n| \le |z_2| < |t| \quad \text{and} \quad |t| + |z_n| \le |t| + |z_2| < 1 \quad \text{for } n \ge 2. \tag{38}$$

So we can go back to the case 1. That means if $t \in D_1$, then $\langle A, B\rangle$ is not discrete.

By simple computation, we see that

$$D_1 \supset \{t\colon 0 < |t| \le \tfrac{1}{2} \quad \text{and} \quad -\pi/4 \le \arg(t) \le \pi/4\}. \tag{39}$$

Now from (33)

$$(A^{-1}M)^2 = \begin{pmatrix} \frac{u^4+1}{u^2-1} & * \\ * & -\frac{u^4+1}{u^2(u^2-1)} \end{pmatrix}. \tag{40}$$

We redefine the matrix B to be

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A^3(A^{-1}M)^2 = \begin{pmatrix} \frac{u^3(u^4+1)}{u^2-1} & * \\ * & -\frac{u^4+1}{u^5(u^2-1)} \end{pmatrix}. \tag{41}$$

Then

$$\operatorname{tr}^2(B) = (a+d)^2 = t^4(t+2), \tag{42}$$

and

$$ad = \frac{-t}{t-2}. \tag{43}$$

If we set $F = A^2$, then from (20)

$$\operatorname{tr}(BFB^{-1}F^{-1}) = \operatorname{tr}(FBF^{-1}B^{-1}) = t^3 + 3t^2 - 2. \tag{44}$$

When $0 < |t| \leq \frac{1}{2}$

$$\text{(45)} \qquad \operatorname{tr}^2(B) \neq 0 \quad \text{and} \quad \operatorname{tr}(BFBF^{-1}) = t^5 + 2t^4 - t^3 - 3t^2 + 2 \neq 2\,.$$

So applying Lemma 1 to B and F yields

$$\text{(46)} \qquad |\operatorname{tr}^2(B)| + |\operatorname{tr}(BFB^{-1}F^{-1}) + 2| \geq 1\,.$$

Thus

$$\text{(47)} \qquad |t^4(t+2)| + |t^3 + 3t^2| \geq 1\,.$$

Let

$$\text{(48)} \qquad D_2 = \{t\colon 0 < |t| \leq \tfrac{1}{2} \quad \text{and} \quad |t^4(t+2)| + |t^3 + 3t^2| < 1\}\,.$$

If $t \in D_2$, then $\langle A, B\rangle$ is not discrete.

By simple computation, we see that

$$\text{(49)} \qquad D_2 \supset \{t\colon 0 < |t| \leq \tfrac{1}{2} \quad \text{and } \pi/4 \leq \arg(t) \leq 7\pi/4\}\,.$$

Combining (39) with (49), we conclude that $\langle A, B\rangle$ is not discrete. So we get (15).

If $\operatorname{tr}(ABAB^{-1}) = 1$, then $z_1 = 0$. The same argument yields $t \notin D_1 \cup D_2$. So $t = 0$ or $|t| > \frac{1}{2}$ and (16) follows. This finishes the proof of the Lemma 2.

4. Conjecture

The results established suggest the following very interesting question: For what kinds of discrete groups $\langle f, g\rangle$ will the left-hand sides of inequalities (1), (3), (5), or (14) be zero?

The left-hand side of (1) is zero if $f(z) = z + 1$, $g(z) = z + \tau$ (τ nonreal). Here f and g are both parabolic and $\langle f, g\rangle$ has signature $(1, 0)$.

The left-hand side of (3) is zero if $f(z) = iz$, $g(z) = (z-1)/(z+1)$. Here f and g are both of order 4 and $\langle f, g\rangle$ has signature $(0, 3;\ 2, 3, 4)$.

The left-hand side of (5) is zero if $f(z) = (z-i)/(z+i)$, $g(z) = -(z-i)/(z+i)$. Here f and g are both of order 3 and $\langle f, g\rangle$ has signature $(0, 3;\ 2, 3, 3)$.

Finally the left-hand side of (14) is zero if $f(z) = -z$, $g(z) = 1/z$. Here f and g are both of order 2 and $\langle f, g\rangle$ has signature $(0, 3;\ 2, 2, 2)$.

In each of the four cases discussed above, the group $\langle f, g\rangle$ is elementary. From these observations, we are led to make the following conjecture:

Let $\langle F, G\rangle$ be an elementary discrete group with $\operatorname{tr}(F) = \operatorname{tr}(G)$. *If $\langle f, g\rangle$ is discrete, then*

$$\text{(50)} \qquad |\operatorname{tr}^2(f) - \operatorname{tr}^2(F)| + |\operatorname{tr}(fgf^{-1}g^{-1}) - \operatorname{tr}(FGF^{-1}G^{-1})| \geq 1\,,$$

provided that some appropriate auxiliary conditions are satisfied.

For example, if we take $F(z) = -z$, $G(z) = -z + 1$, the signature of $\langle F, G\rangle$ is $(0, 3;\ 2, 2, \infty)$. Then (50) will become

$$\text{(51)} \qquad |\operatorname{tr}^2(f)| + |\operatorname{tr}(fgf^{-1}g^{-1}) - 2| \geq 1\,.$$

So far we can prove only the following: [10] *If* $\langle f, g\rangle$ *is nonelementary discrete and* $\operatorname{tr}(f) \neq 0$, *then*

$$|\operatorname{tr}^2(f)| + |\operatorname{tr}(fgf^{-1}g^{-1}) - 2| > 2(\sqrt{2} - 1) = 0.828\cdots. \tag{52}$$

The accurate lower bound of (52) is unknown.

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