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A WEAK-TYPE ESTIMATE FOR FOURIER-BESSEL MULTIPLIERS

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ABSTRACT. We apply Hörmander's technique to prove a weak-type $(1,1)$ estimate for multiplier operators with respect to the Fourier-Bessel transform. This improves a result in [4, 5].

1. INTRODUCTION

Consider the measure space $(\mathbf{R}^+, d\mu(x))$ where $d\mu(x) = x^r dx$, r being a fixed positive real number. By $L^p(\mu)$, $1 \le p \le \infty$, we denote the corresponding Lebesgue spaces equipped with the norms

$$\|f\|_p = \left(\int_0^\infty |f|^p \, d\mu\right)^{1/p}.$$

Let $\hat{f}(\lambda)$, $\lambda \ge 0$, denote the Fourier-Bessel transform of the function $f \in L^1(\mu)$. Explicitly this means

$$\hat{f}(\lambda) = \int_0^\infty f(x)\phi_\lambda(x)\, d\mu(x),$$

where $\phi_\lambda(x) = a(r)(\lambda x)^{-(r-1)/2} J_{(r-1)/2}(\lambda x)$, $x \ge 0$, $a(r) = 2^{(r-1)/2}\Gamma((r+1)/2)$, and J_α denotes the Bessel function of the first kind of order α. The functions ϕ_λ, $\lambda \ge 0$, are eigenfunctions of the second order Bessel differential operator

$$L = -\left(\frac{d}{dx^2} + \frac{r}{x}\frac{d}{dx}\right).$$

More precisely, we have

$$L\phi_\lambda = \lambda^2 \phi_\lambda. \tag{1.1}$$

If $r = n-1$, $n \ge 2$ being an integer, then

$$\hat{f}(\|\overline{\lambda}\|) = \frac{1}{2}\pi^{-n/2}\Gamma\left(\frac{n}{2}\right)\int_{\mathbf{R}^n} f(\|\overline{x}\|)\exp(-i\langle\overline{\lambda},\overline{x}\rangle)\, d\overline{x}$$

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for $\bar{\lambda} \in \mathbf{R}^n$. Thus $\hat{f}$ gives the radial part of the Fourier transform of the radial function $f(\|\bar{x}\|)$ on $\mathbf{R}^n$ (observe that $f \in L^1(\mu)$ if and only if $f(\|\bar{x}\|) \in L^1(\mathbf{R}^n, d\bar{x})$ where $\|\cdot\|$ denotes the Euclidean norm and $d\bar{x}$ is Lebesgue measure on $\mathbf{R}^n$). In the case $r = n-1$, one can easily see that for the Fourier-Bessel transform, both the inversion theorem and Plancherel's formula hold. In particular we have

$$f(x) = a(r)^{-1} \int_0^\infty \hat{f}(\lambda)\phi_\lambda(x)\, d\mu(\lambda)$$

almost everywhere on $\mathbf{R}^+$ providing $f \in L^1(\mu)$ and $\hat{f} \in L^1(\mu)$. For $f \in L^2(\mu)$ we have

$$\int_0^\infty |f(x)|^2\, d\mu(x) = a(r)^{-2} \int_0^\infty |\hat{f}(\lambda)|^2\, d\mu(\lambda).$$

The above results for general $r > 0$ can be established via a summability argument similar to that in the case of the ordinary Fourier transform. The key idea is that the Gaussian is invariant under the Fourier-Bessel transform when appropriately normalized. We also note that in the case $r = n-1$, the operator $-L$ is the radial part of the ordinary Laplace operator on $\mathbf{R}^n$. Finally, we mention that C will be used to denote a constant which may vary from line to line.

For any bounded function m on $(0,\infty)$ we define the multiplier operator T_m by $(T_m f)\hat{} = m\hat{f}$. The aim of this note is to prove the following theorem which is a generalization of the celebrated Hörmander-Mihlin multiplier theorem for the Fourier transform (see [3]).

Theorem 1.1. *Let k denote the least even integer $> (r+1)/2$ and let $m \in C^k(0,\infty)$ be a bounded function which satisfies*

$$\left(\int_{R/2}^{R} |m^{(s)}(\lambda)|^2\, d\mu(\lambda))\right)^{1/2} \le BR^{(r+1)/2-s}, \tag{1.2}$$

where B is a constant independent of $R > 0$ and $s = 0, 1, \ldots, k$. Then the operator T_m is of weak-type $(1,1)$, i.e.

$$\mu(\{x \in (0,\infty) : |T_m f(x)| > \alpha\}) \le C\alpha^{-1}\|f\|_1 \tag{1.3}$$

where C is independent of $\alpha > 0$ and $f \in L^1(\mu)$.

Using the interpolation theorem of Marcinkiewicz and duality we also obtain the following

Corollary 1.2. *Let m satisfy the assumptions of the Theorem 1.1. Then for $1 < p < \infty$, T_m is of strong type (p,p), i.e. $\|T_m f\|_p \le C_p \|f\|_p$.*

Remark. The result of the above corollary was obtained previously by the second author by means of a Littlewood-Paley theory for the Fourier-Bessel transform (see [4, 5]). The proof of the above theorem is more direct and is based upon a generalized convolution structure which is natural for the Fourier-Bessel

transform. The novel ingredient of our proof is a Bernstein-type inequality for this convolution structure. We establish this inequality in the following section.

2. A Bernstein-type inequality for generalized translations

Since Hirschman and Bochner considered the Banach algebra of radial functions over thirty years ago, it has been known that $L^1(\mu)$ admits a commutative Banach algebra structure for which the Fourier-Bessel transform becomes the Gelfand transform (see [4 and 5] for details and for other references). In order to describe this structure we define the generalized translation operator T^y, $y \geq 0$, for suitable functions f, by

$$T^y f(x) = \int_0^\pi f((x,y)_\theta)\, d\nu(\theta) \tag{2.1}$$

where $(x,y)_\theta = (x^2 + y^2 - 2xy\cos\theta)^{1/2}$, $0 \leq \theta \leq \pi$, $x, y \geq 0$ and $d\nu(\theta)$ denotes the probability measure $b(r)(\sin\theta)^{r-1}\, d\theta$ on $[0,\pi]$. Here

$$b(r) = \pi^{-1/2}\Gamma((r+1)/2)\Gamma(r/2)^{-1}.$$

The operator T^y can also be described by

$$T^y f(x) = \int_{|x-y|}^{x+y} f(z)\, dW_{x,y}(z) \tag{2.2}$$

where the probability measure $dW_{x,y}(z)$ is supported on $[|x-y|, x+y]$ and given by

$$dW_{x,y}(z) = c(r)\frac{\Delta(x,y,z)^{r-2}}{(xyz)^{r-1}}\, d\mu(z),$$

where $c(r) = 2^{r-2}\Gamma((r+1)/2)\Gamma(r/2)^{-1}\pi^{-1/2}$. In the above formula $x, y, z \geq 0$, $|x-y| \leq z \leq x+y$, and $\Delta(x,y,z)$ denotes the area of a triangle with sides x, y, z. It is quite straightforward to go from (2.2) to (2.1) by a change of variables. Next, it is easy to check that the measure $dW_{x,y}(z)\, d\mu(y)$ is symmetric with respect to y and z, i.e. $dW_{x,y}(z)\, d\mu(y) = dW_{x,z}(y)\, d\mu(z)$. From this it follows that the operators T^y, $y \geq 0$, are selfadjoint on $L^2(\mu)$ and moreover that

$$\int_0^\infty f(x)T^y g(x)\, d\mu(x) = \int_0^\infty T^y f(x) g(x)\, d\mu(x), \tag{2.3}$$

for any reasonable pair of functions f and g, e.g. if $f \in L^\infty(\mu)$ and $g \in L^1(\mu)$. It is also easy to see that generalized translations are submarkovian contractions, i.e. $0 \leq f \leq 1$ implies $0 \leq T^y f \leq 1$ and $\|T^y f\|_p \leq \|f\|_p$, for $1 \leq p \leq \infty$. We now define the generalized convolution $f * g$ for $f, g \in L^1(\mu)$ by

$$f * g(x) = \int_0^\infty f(y)T^y g(x)\, d\mu(y). \tag{2.4}$$

It is well known that the following identity involving the functions ϕ_λ and the generalized translations T^y is valid

$$T^y\phi_\lambda(x) = \phi_\lambda(x)\phi_\lambda(y) \tag{2.5}$$

(see [4, 5] for references). Using (2.3) and (2.5) one can easily verify that for any $f, g \in L^1(\mu)$ we have $(f * g)\hat{} = \hat{f}\hat{g}$ and $(T^y f)\hat{}(\lambda) = \phi_\lambda(y)\hat{f}(\lambda)$. Therefore

$$T^y g * f = g * T^y f. \tag{2.6}$$

It is also easy to check that with the pseudodistance $\rho(x, y) = |x - y|$, the space $(\mathbf{R}^+, \mu, \rho)$ becomes a space of homogeneous type in the sense of [1]. Finally, from (2.2) it is immediate that $T^y f(x) = 0$ if f vanishes on the interval $(|x - y|, x + y)$ and that $|T^y f(x)| \le T^y(|f|)(x)$ for $x, y \ge 0$.

In the proof of the Hörmander-Mihlin theorem (cf. [3]), Bernstein's inequality plays an important role. We now prove a Bernstein-type inequality for the generalized translations T^y, $y \ge 0$.

Theorem 2.1. *Suppose $h \in L^1(\mu)$ is a differentiable function on $\mathbf{R}^+$ with $h'(x) \in L^1(\mu)$. Then*

$$\|T^{y_1}h - T^{y_2}h\|_1 \le \|h'\|_1 |y_1 - y_2|. \tag{2.7}$$

Proof. Using (2.1) we write

(2.8)

$$\begin{aligned}\|T^{y_1}h - T^{y_2}h\|_1 &= \int_0^\infty \left| \int_0^\pi [h((x, y_1)_\theta) - h((x, y_2)_\theta)]\, d\nu(\theta) \right| d\mu(x) \\ &\le \int_0^\infty \int_0^\pi |h((x, y_1)_\theta) - h((x, y_2)_\theta)|\, d\nu(\theta)\, d\mu(x) \\ &= \int_0^\infty \int_0^\pi \left| \int_0^1 \frac{d}{ds}(h((x, y_2 + s(y_1 - y_2))_\theta))\, ds \right| d\nu(\theta)\, d\mu(x).\end{aligned}$$

For fixed $x, y_1, y_2 \ge 0$ and $\theta \in (0, \pi]$, let

$$\Phi(s) \equiv \Phi_{x,\theta}^{y_1, y_2}(s) = (x, y_2 + s(y_1 - y_2))_\theta, \qquad s \in [0, 1].$$

Then

$$\left| \frac{d}{ds}\Phi(s) \right| \le |y_1 - y_2| \tag{2.9}$$

independently of x, y_1, y_2, θ. To verify (2.9) consider the vectors $X = (x, 0)$, $Y_i = (y_i \cos\theta, y_i \sin\theta)$, $i = 1, 2$, in $\mathbf{R}^2$. Letting $\|\cdot\|$ denote the Euclidean norm on $\mathbf{R}^2$ it is easy to see that

$$(x, y_2 + s(y_1 - y_2))_\theta = \|Y_2 + s(Y_1 - Y_2) - X\| \equiv \|\xi_s\|.$$

Since $\nabla(\|\cdot\|)(\xi) = \xi/\|\xi\|$ for $\xi \in \mathbf{R}^2$, Schwarz' inequality implies

$$\begin{aligned}\left| \frac{d}{ds}\Phi(s) \right| &= \left| \frac{d}{ds}(\|Y_2 + s(Y_1 - Y_2) - X\|) \right| = \left| \left\langle \frac{\xi_s}{\|\xi_s\|}, Y_1 - Y_2 \right\rangle \right| \\ &\le \|Y_1 - Y_2\| = |y_1 - y_2|.\end{aligned}$$

Here $\langle\cdot,\cdot\rangle$ denotes the ordinary Euclidean inner product on $\mathbf{R}^2$. This gives (2.9). Now, using (2.9) and the contraction property of generalized translations, we have from (2.8)

$$\begin{aligned}
&\|T^{y_1}h - T^{y_2}h\|_1 \\
&\quad \le |y_1 - y_2| \int_0^1 \int_0^\infty \int_0^\pi |h'((x, y_2 + s(y_1 - y_2))_\theta)|\, d\nu(\theta)\, d\mu(x)\, ds \\
&\quad \le |y_1 - y_2| \int_0^1 \|T^{y_1 + s(y_1 - y_2)}(|h'|)\|_1\, ds \\
&\quad \le |y_1 - y_2| \|h'\|_1 .
\end{aligned}$$

This completes the proof of Theorem 2.1.

As in the case of the Fourier transform (cf. [2]) Theorem 2.1 implies the following corollary. We include its proof for the sake of completeness.

Corollary 2.2. *There exists a constant* $C > 0$ *such that for any* $\lambda > 0$ *and any function* $f \in L^1(\mu)$ *with* $\operatorname{supp} \hat{f} \subset (0, \lambda)$

$$\|T^{y_1}f - T^{y_2}f\|_1 \le C\lambda |y_1 - y_2| \|f\|_1 . \tag{2.10}$$

Proof. Choose $\chi \in C^\infty(\mathbf{R}^+)$ such that $\chi(\lambda) = 1$ for $0 < \lambda \le 1$ and $\chi(\lambda) = 0$ for $\lambda \ge 2$. Then by the inversion theorem we have $\chi = \hat{h}$ for some C^∞ function $h \in L^1(\mu)$ with $h' \in L^1(\mu)$. Let $h_\varepsilon(x) = \varepsilon h(\varepsilon x)$ for $\varepsilon > 0$. Then $\hat{h}_\varepsilon(\lambda) = \chi(\lambda/\varepsilon) = 1$ for $0 \le \lambda \le \varepsilon$. Thus, for $\varepsilon = \lambda$ we have

$$T^{y_1}f - T^{y_2}f = h_\lambda * (T^{y_1}f - T^{y_2}f) = (T^{y_1}h_\lambda - T^{y_2}h_\lambda) * f$$

and therefore by (2.7) we have

$$\begin{aligned}
\|T^{y_1}f - T^{y_2}f\|_1 &\le \|T^{y_1}h_\lambda - T^{y_2}h_\lambda\|_1 \|f\|_1 = \|T^{\lambda y_1}h - T^{\lambda y_2}h\|_1 \|f\|_1 \\
&\le \lambda |y_1 - y_2| \|h'\|_1 \|f\|_1 .
\end{aligned}$$

This establishes (2.10) with $C = \|h'\|_1$ and completes the proof of the corollary.

3. Proof of Theorem 1.1

We closely follow the original proof of Hörmander (see [3, pp. 121–123]). Choose a function $\psi \in C_0^\infty(\mathbf{R}^+)$ with support in $(\frac{1}{2}, 2)$ such that $\sum_{-\infty}^{\infty} \psi(2^{-j}\lambda) = 1$, $\lambda > 0$, and let

$$m_j(\lambda) = m(\lambda)\psi(2^{-j}\lambda) \equiv m(\lambda)\psi_j(\lambda) \tag{3.1}$$

and

$$k_j(x) = \check{m}_j(x) = a(r)^{-1} \int_0^\infty m_j(\lambda)\phi_\lambda(x)\lambda^r\, d\lambda . \tag{3.2}$$

Then $T_m = \sum_{-\infty}^{\infty} T_{m_j}$ where $T_{m_j}f = k_j * f$ and $k_j \in L^1(\mu)$. In order to prove the weak-type $(1,1)$ inequality (1.3), it suffices to establish (see [1, p. 75])

$$\sum_{j=-\infty}^{\infty} \int_{|x-y_0|>2|y-y_0|} |T^y k_j(x) - T^{y_0}k_j(x)|\, d\mu(x) \le C , \tag{3.3}$$

where C is independent of y, $y_0 \geq 0$. This will be implied in a standard way by the following estimates:

$$\int_{|x-y_0|\geq 2|y-y_0|} |T^y k_j(x) - T^{y_0} k_j(x)|\, d\mu(x) \leq C(2^j|y-y_0|)^{(r+1)/2-k}, \tag{3.4}$$

which is good when $2^j|y-y_0| \geq 1$, and

$$\int_{|x-y_0|\geq 2|y-y_0|} |T^y k_j(x) - T^{y_0} k_j(x)|\, d\mu(x) \leq C2^j|y-y_0|, \tag{3.5}$$

which is good when $2^j|y-y_0| < 1$.

To establish (3.4) we first note that conditions (1.2) imply

$$\left(\int_0^\infty |L^l m_j(\lambda)|^2\, d\mu(\lambda)\right)^{1/2} \leq CB(2^j)^{(r+1)/2-2l}, \qquad l = 0, 1, \dots, k/2. \tag{3.6}$$

In fact, by induction, one can easily verify the following Leibniz rule for L: for $f, g \in C^{2l}(\mathbf{R}^+)$

$$L^l(fg) = \sum_{0<\alpha+\beta\leq 2l} C^l_{\alpha,\beta} f^{(\alpha)} g^{(\beta)} \lambda^{\alpha+\beta-2l}.$$

Thus, taking $R = 2^j$ in (1.2), we can estimate the left-hand side of (3.6) by

$$\begin{aligned}
&C \sum_{0<\alpha+\beta\leq 2l} \left(\int_0^\infty |m^{(\alpha)}(\lambda)\psi_j^{(\beta)}(\lambda)\lambda^{\alpha+\beta-2l}|^2\, d\mu(\lambda)\right)^{1/2} \\
&\quad \leq C \sum_{0<\alpha+\beta\leq 2l} 2^{-\beta j} \left(\int_{2^{j-1}}^{2^{j+1}} |m^{(\alpha)}(\lambda)\lambda^{\alpha+\beta-2l}|^2\, d\mu(\lambda)\right)^{1/2} \\
&\quad \leq C \sum_{0<\alpha+\beta\leq 2l} (2^j)^{\alpha-2l} \left(\int_{2^{j-1}}^{2^{j+1}} |m^{(\alpha)}(\lambda)|^2\, d\mu(\lambda)\right)^{1/2} \\
&\quad \leq CB(2^j)^{(r+1)/2-2l}.
\end{aligned}$$

This gives (3.6). Next, using the Plancherel theorem and (3.6) we obtain

$$\begin{aligned}
\|(1+(2^jx)^2)^{k/2} k_j\|_2 &= \|[(1+2^{2j}L)^{k/2} m_j]^\vee\|_2 \\
&= a(r)^{-2} \|(1+2^{2j}L)^{k/2} m_j\|_2 \\
&\leq C \sum_{l=0}^{k/2} (2^{2j})^l \|L^l m_j\|_2 \leq CB(2^j)^{(r+1)/2}.
\end{aligned}$$

Thus by Schwarz' inequality and the estimate $\|(1+(2^jx)^2)^{-k/2}\|_2 \leq C(2^j)^{-(r+1)/2}$ we obtain

$$\begin{aligned}
\|k_j\|_1 &\leq \|(1+(2^jx)^2)^{k/2} k_j\|_2 \|(1+(2^jx)^2)^{-k/2}\|_2 \\
&\leq CB.
\end{aligned} \tag{3.7}$$

Moreover, since $\|(2^j x)^k k_j\|_2 \le \|(1+(2^j x)^2)^{k/2} k_j\|_2$ we obtain in the same way

$$\int_{x\ge t} |k_j(x)|\, d\mu(x) \le \|(2^j x)^k k_j\|_2 \left(\int_{x\ge t} (2^j x)^{-2k}\, d\mu(x)\right)^{1/2} \le CB(2^j t)^{(r+1)/2-k}. \tag{3.8}$$

If χ_A is the characteristic function of a set $A \subset \mathbf{R}^+$, then $0 \le T^y \chi_A(x) \le 1$ for $x, y \in \mathbf{R}^+$. We claim that

$$T^{y_0} \chi_{\{z:\ |z-y_0|>2|y-y_0|\}} \le \chi_{\{z:\ z>2|y-y_0|\}} \tag{3.9}$$

and

$$T^{y} \chi_{\{z:\ z-y_0|>2|y-y_0|\}} \le \chi_{\{z;z>|y-y_0|\}}. \tag{3.10}$$

We only prove (3.9) and note that the proof of (3.10) is similar. Let E denote the set $\{z: |z-y_0| > 2|y-y_0|\}$. To verify (3.9) we need only check that $0 \le x \le 2|y-y_0|$ implies $T^{y_0}\chi_E(x) = 0$. Since $0 \le x \le 2|y-y_0|$ implies $(|y_0-x|, y_0+x) \cap E = \varnothing$, it follows that $T^{y_0}\chi_E(x) = 0$ (see remark after (2.6)). This proves (3.9).

Using (3.7) through (3.10) and the selfadjointness of generalized translations, cf. (2.3), we find that the left-hand side of (3.4) is majorized by

$$\begin{aligned}\int_0^\infty \chi_E(x)|T^y k_j(x)|\, d\mu(x) &+ \int_0^\infty \chi_E(x)|T^{y_0} k_j(x)|\, d\mu(x)\\ &\le \int_{|y-y_0|}^\infty |k_j(x)|\, d\mu(x) + \int_{2|y-y_0|}^\infty |k_j(x)|\, d\mu(x)\\ &\le CB(2^j|y-y_0|)^{(r+1)/2-k}.\end{aligned}$$

Thus (3.4) is established.

Finally, to establish (3.5) we use Bernstein's inequality. Since $\hat{k}_j = m_j$ has support in $(0, 2^{j+1})$ it follows from Corollary (2.2) and (3.7) that

$$\begin{aligned}\int_{|x-y_0|\ge 2|y-y_0|} |T^y k_j - T^{y_0} k_j|\, d\mu &\le \|T^y k_j - T^{y_0} k_j\|_1\\ &\le 2^{j+1}|y-y_0| \|k_j\|_1 \le CB2^j|y-y_0|.\end{aligned}$$

Thus (3.5) is established. Combining (3.4) and (3.5) we obtain (3.3) which completes the proof of the theorem.

4. A final remark

In connection with the Bernstein-type inequality (2.7) the following question arises. Consider the "mean value" operators T^y, $y \ge 0$, defined for functions on $\mathbf{R}^n$, $n \ge 2$, by

$$T^y f(\overline{x}) = \int_{S^{n-1}} f(\overline{x} + y\overline{t})\, d\sigma(\overline{t}),$$

where $d\sigma$ is normalized surface measure on the sphere S^{n-1}. In other words T^y is the convolution operator with the uniformly distributed probability measure on $\{\overline{x} \in \mathbf{R}^n : \|\overline{x}\| = y\}$. Is it true that for any $h \in C^1(\mathbf{R}^n)$ with, say, $\|\nabla h\| \in L^1(\mathbf{R}^n)$, the following estimate holds

$$\|T^{y_1}h - T^{y_2}h\|_{L^1(\mathbf{R}^n)} \leq C|y_1 - y_2|,$$

where C depends on h but not on y_1 and y_2? We note that Theorem 2.1 gives a positive answer to this problem only in the case when h is radial.

But one can rather easily remark that the above problem may be answered affirmatively by writing out $\|T^{y_1}h - T^{y_2}h\|_{L^1(\mathbf{R}^n)}$, interchanging the order of integration and then applying the ordinary Bernstein-type inequality on $\mathbf{R}^n$. This yields an easy proof of Theorem 2.1 in the case of integral values of r.

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