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AN EXTENSION OF RELLICH'S INEQUALITY

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ABSTRACT. In this paper we present a theorem which extends the results of an inequality originally due to Franz Rellich [4]. The theorem by Rellich establishes an inequality widely used in the spectral theory of partial differential operators. Our theorem allows for a broader range of application by extending the class of functions to which the theorem is applicable. Many authors call upon inequalities similar to the one established in our theorem in dealing with problems concerning essential self-adjointness of Schrödinger operators and other problems arising in oscillation theory of elliptic operators.

In the first part of the paper we present Rellich's inequality and discuss some problems dealing with symmetric operators on Hilbert spaces where Rellich's inequality is a useful tool.

We shall also discuss some important extensions of Rellich's work which were established by other mathematicians. One such extension was proved by W. Allegretto [1] in dealing with elliptic equations of order $2n$. Another extension was established by U. W. Schmincke [5] in considering essential self-adjointness criteria of Schrödinger operators. Schmincke's extension is of particular interest to us due to his elegant proof. We follow Schmincke's method of proof.

We then state and prove our generalization of Rellich's inequality along with a useful corollary.

The paper concludes with a few brief comments on our result and other work which could be done with Rellich's inequality.

RELLICH'S INEQUALITY

Define the operator T by

$$Tu = \Delta\Delta u - c|x|^{-4}u$$

on $L^2(\mathbf{R}^n)$ with domain $\mathscr{D}(T) = C_0^\infty(\mathbf{R}^n\backslash\{0\})$. Then T is symmetric on $\mathscr{D}(T)$. As Rellich mentions in [4] his inequality can be viewed as a means of finding the largest positive real number c such that the eigenvalues for the Friedrich's extension of T are nonnegative. Also, this could be viewed as finding the least point in the spectrum of $\Delta\Delta u$, $u \in C_0^\infty(\mathbf{R}^n\backslash\{0\})$, in the weighted

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L^2-space $L^2_\omega(\mathbf{R}^n)$ with $\omega = |x|^{-4}$. For the solution we turn to Rellich's inequality.

Rellich's inequality [4]. *Suppose that* $u(x)$ *is a function in* $C_0^\infty(\mathbf{R}^n\backslash\{0\})$ *which is not identically zero, then*

$$\int_{\mathbf{R}^n} |\Delta u|^2\,dx \geq \frac{n^2(n-4)^2}{16}\int_{\mathbf{R}^n} |u|^2|x|^{-4}\,dx, \qquad n \neq 2.$$

This inequality also holds for $n = 2$ provided that the following additional hypotheses are satisfied:

$$\int_0^{2\pi} u(|x|\cos\theta, |x|\sin\theta)\cos\theta\,d\theta = 0,$$

$$\int_0^{2\pi} u(|x|\cos\theta, |x|\sin\theta)\sin\theta\,d\theta = 0, \qquad 0 < |x| < \infty.$$

Rellich also proved that for each set of three constants c, a, r, such that $r > 0$ and $c > 0$ for $n = 2$, $r > 0$ and $c > n^2(n-4)^2/16$ for $n \neq 2$, there exists a function $u_0 \in C_0^\infty(\mathbf{R}^n\backslash\{0\})$ which vanishes identically for $|x| > r$ and a function $u_\infty \in C_0^\infty(\mathbf{R}^n\backslash\{0\})$ which vanishes identically for $|x| < r$ satisfying the inequalities

$$c\int_{0<|x|<\infty} |x|^{-4}|u_0|^2\,dx > \int_{0<|x|<\infty} |\Delta u_0|^2\,dx + a\int_{0<|x|<\infty} |u_0|^2\,dx,$$

$$c\int_{0<|x|<\infty} |x|^{-4}|u_\infty|^2\,dx > \int_{0<|x|<\infty} |\Delta u_\infty|^2\,dx.$$

Since the proof of the theorem, as given by Rellich, is quite lengthy, we shall not state it here.

Hence, for the operator T given above, $c = n^2(n-4)^2/16$ is the largest real number such that the operator T is nonnegative.

Extensions of Rellich's inequality

In obtaining several nonoscillation theorems for elliptic equations of order $2n$, W. Allegretto extended several well-known nonoscillation theorems for elliptic equations of order 2 and 4. In that paper, Allegretto needed Rellich's inequality to include powers of $|x|$. (In our statement of Allegretto's lemma we have replaced α with $\alpha + 4$ to simplify comparison with our own result.)

Lemma 1 (Allegretto [1]). *Let* $\theta \in C_0^\infty(\mathbf{R}^n\backslash\{0\})$, $\alpha \in \mathbf{R}^1$, $\alpha \leq -4$. *Then the following inequality is valid*:

$$\int_{\mathbf{R}^n} |x|^{\alpha+4}(\Delta\theta)^2\,dx \geq K(\alpha)\int_{\mathbf{R}^n} |x|^\alpha\theta^2\,dx,$$

where

$$K(\alpha) = \frac{(\alpha+n)^2(\alpha+4-n)^2}{16} + \tau(\alpha),$$

$$\tau(\alpha) = \inf_{k\in\{0,1,\dots\}}\{k(k+n-2)[k^2 + (n-2)k + \tfrac{1}{2}(n^2 - 4n - \alpha^2 - 4\alpha)]\}.$$

This inequality shows that the equation

$$-\Delta(|x|^{\alpha+4}\Delta\theta) - K(\alpha)|x|^{\alpha}\theta = 0$$

is nonoscillatory at ∞.

For more details concerning oscillation theory of elliptic equations we refer the reader to Allegretto [1], Kreith [2], and Swanson [6].

In exploring self-adjointness criteria for a Schrödinger operator, U. W. Schmincke proved a lemma which is a form of Rellich's inequality. Let $\mathbf{R}_+^n$ denote $\mathbf{R}^n\backslash\{0\}$, $n \geq 2$.

Lemma 2 (Schmincke [5]). *Suppose* $u \in C_0^\infty(\mathbf{R}_+^n)$ *and* $s \in [-n(n-4)/2, \infty)$. *Then*

$$\int_{\mathbf{R}_+^n} |\Delta u|^2\, dx \geq -s\int_{\mathbf{R}_+^n} |\nabla u|^2|x|^{-2}\, dx + \frac{(n-4)^2}{16}(n^2+4s)\int_{\mathbf{R}_+^n} |u|^2|x|^{-4}\, dx.$$

In the proof Schmincke uses the Hardy-type (or Friedrichs) inequality:

$$4\int_{\mathbf{R}_+^n} |\nabla u|^2|x|^{-2}\, dx \geq (n-4)^2\int_{\mathbf{R}_+^n} |u|^2|x|^{-4}\, dx,$$

where $u \in C_0^\infty(\mathbf{R}_+^n)$. The proof of our theorem is very similar but we use the Friedrichs inequality [3]:

$$4\int_{\mathbf{R}^n} |\nabla u|^2|\nabla g|^2|\Delta g|^{-1}\, dx \geq \int_{\mathbf{R}^n} |u|^2|\Delta g|\, dx,$$

where $u \in C_0^\infty(\mathbf{R}^n)$, $g \in C^2(\mathbf{R}^n)$, and $\Delta g \neq 0$, which allows us to achieve more general results.

Schmincke was able to retrieve Rellich's inequality except for $n \leq 3$. This was due to the requirement that $s \in [-n(n-4)/2, \infty)$, since for $n \leq 3$, s cannot take the value zero. However, a second-order term is included which was not the case with Rellich's inequality.

Throughout the remainder of this paper we assume that Ω is an open, connected subset of $\mathbf{R}^n$ that is not necessarily bounded, and that the boundary of Ω, $\partial\Omega$, is sufficiently smooth in order that the first formula of Green applies. Denote the unit outward normal by $\vec{n}$. Let $\partial\Omega^+$ be the portion of $\partial\Omega$ where $x\cdot\vec{n} > 0$, $\partial\Omega^-$ the portion of $\partial\Omega$ where $x\cdot\vec{n} < 0$.

We need the following lemma.

Lemma 3 (Lewis [3]). *Let* $g \in C^2(\Omega)$ *be real-valued and satisfy* $\Delta g(x) \neq 0$ *on* Ω; *then*

$$\int_\Omega |\Delta g(x)||\theta(x)|^2\, dx \leq 2\int_\Omega |\nabla g(x)||\nabla\theta(x)||\theta(x)|\, dx \leq 4\int_\Omega |\Delta g(x)|^{-1}|\nabla g(x)|^2|\nabla\theta(x)|^2\, dx$$

for all $\theta(x) \in C_0^\infty(\mathbf{R}^n)$ *that satisfy*

$$(-1)^\eta \int_{\partial\Omega} \nabla g(x) \cdot \vec{n} |\theta(s)|^2 \, ds \leq 0,$$

where $\eta = 0$ *if* $\Delta g > 0$ *on* Ω *and* $\eta = 1$ *if* $\Delta g < 0$ *on* Ω.

Note that this lemma applies for all $\theta \in C_0^\infty(\Omega)$ since for such θ,

$$(-1)^\eta \int_{\partial\Omega} \nabla g \cdot \vec{n} |\theta(s)|^2 \, ds = 0.$$

Corollary 4 (Lewis [3]). *If* $\theta \in C_0^1(\Omega \backslash \{0\})$, *then*

$$4 \int_\Omega |x|^\beta |\nabla \theta(x)|^2 \, dx \geq (\beta - 2 + n)^2 \int_\Omega |x|^{\beta - 2} |\theta(x)|^2 \, dx. \tag{1}$$

Moreover, for $n \geq 2$, *inequality* (1) *is valid for all* $\theta \in \{u \in C_0^1(\mathbf{R}_+^n) | u(x) = 0$ *on* $\partial\Omega^+\}$ *when* $\beta > 2 - n$; *when* $\beta < 2 - n$, *it is valid for all* $\theta \in \{u \in C_0^1(\mathbf{R}_+^n) | u(x) = 0$ *on* $\partial\Omega^-\}$.

We now state and prove our main result.

Theorem 5. *If* $g \in C^2(\Omega)$ *and* $\Delta g \neq 0$ *in* Ω, *then for any* $\delta \geq 0$, $\varepsilon > 0$, *and all* $u \in C_0^\infty(\Omega)$

$$\int_\Omega |g|^2 |\Delta g|^{-1} |\Delta u|^2 \, dx \geq -\varepsilon \int_\Omega (2g\Delta g + \delta |\nabla g|^2) |\Delta g|^{-1} |\nabla u|^2 \, dx + \varepsilon \left(1 - \varepsilon + \frac{\delta}{4}\right) \int_\Omega |\Delta g| |u|^2 \, dx. \tag{2}$$

Proof. Suppose $u \in C_0^\infty(\Omega)$, $g \in C^2(\Omega)$, and $\Delta g \neq 0$. We can rewrite (2) as

$$\begin{aligned} \int_\Omega |g|^2 |\Delta g|^{-1} |\Delta u|^2 \, dx & \\ \geq -2\varepsilon \operatorname{sgn}(\Delta g) \int_\Omega g |\nabla u|^2 \, dx &- \varepsilon\delta \int_\Omega |\nabla g|^2 |\Delta g|^{-1} |\nabla u|^2 \, dx \\ + \varepsilon \left(1 - \varepsilon + \frac{\delta}{4}\right) \int_\Omega |\Delta g| |u|^2 \, dx&, \end{aligned} \tag{3}$$

where $\operatorname{sgn}(\Delta g) = \Delta g / |\Delta g|$.

Let A, B, C, D denote the integrals (without the exterior constants) in (3) successively.

First we must show that

$$D = \operatorname{sgn}(\Delta g) \int_\Omega \Delta(|u|^2) g \, dx.$$

That is,

$$\int_\Omega |u|^2 |\Delta g| \, dx = \operatorname{sgn}(\Delta g) \int_\Omega \Delta(|u|^2) g \, dx.$$

Applying Green's first formula to $\int_\Omega |u|^2 \Delta g \, dx$ we obtain

$$\int_\Omega |u|^2 \Delta g \, dx = -\int_\Omega \nabla(|u|^2) \nabla g \, dx,$$

since $u \in C_0^\infty(\Omega)$. Applying Green's first formula again we have

$$\int_\Omega |u|^2 \Delta g\,dx = \int_\Omega \Delta(|u|^2) g\,dx$$

thus

$$\int_\Omega |u|^2 |\Delta g|\,dx = \operatorname{sgn}(\Delta g) \int_\Omega \Delta(|u|^2) g\,dx,$$

which is what we wished to show. Since $\Delta(|u|^2) = 2[\operatorname{Re}(u\Delta\bar{u}) + |\nabla u|^2]$, we can write

$$\begin{aligned} D &= \operatorname{sgn}(\Delta g) \int_\Omega \Delta(|u|^2) g\,dx = 2\operatorname{sgn}(\Delta g) \int_\Omega [\operatorname{Re}(u\Delta\bar{u}) + |\nabla u|^2] g\,dx \\ &= 2\operatorname{sgn}(\Delta g) \int_\Omega \operatorname{Re}(u\Delta\bar{u}) g\,dx + 2\operatorname{sgn}(\Delta g) \int_\Omega |\nabla u|^2 g\,dx \\ &\le \left|2\operatorname{sgn}(\Delta g) \int_\Omega \operatorname{Re}(u\Delta\bar{u}) g\,dx\right| + 2\operatorname{sgn}(\Delta g) \int_\Omega |\nabla u|^2 g\,dx \\ &\le 2\int_\Omega |u||\Delta\bar{u}||g|\,dx + 2\operatorname{sgn}(\Delta g) \int_\Omega |\nabla u|^2 g\,dx \\ &= 2\int_\Omega |u||\Delta g|^{1/2}|g||\Delta u||\Delta g|^{-1/2}\,dx + 2\operatorname{sgn}(\Delta g) \int_\Omega |\nabla u|^2 g\,dx. \end{aligned}$$

We now apply the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} D &= \operatorname{sgn}(\Delta g) \int_\Omega \Delta(|u|^2) g\,dx \\ &\le 2\left[\int_\Omega |u|^2|\Delta g|\,dx\right]^{1/2} \left[\int_\Omega |g|^2|\Delta u|^2|\Delta g|^{-1}\,dx\right]^{1/2} \\ &\quad + 2\operatorname{sgn}(\Delta g) \int_\Omega g|\nabla u|^2\,dx. \end{aligned}$$

That is,

$$\begin{aligned} 2\int_\Omega &\operatorname{sgn}(\Delta g) g|u|\Delta|u|\,dx + 2\operatorname{sgn}(\Delta g)\int_\Omega g|\nabla u|^2\,dx \\ &\le 2\left[\int_\Omega |u|^2|\Delta g|\,dx\right]^{1/2} \left[\int_\Omega |g|^2|\Delta g|^{-1}|\Delta u|^2\,dx\right]^{1/2} \\ &\quad + 2\operatorname{sgn}(\Delta g) \int_\Omega g|\nabla u|^2\,dx, \end{aligned}$$

i.e., $D \le 2D^{1/2}A^{1/2} + 2\operatorname{sgn}(\Delta g)B$.

Since $2D^{1/2}A^{1/2} \le \varepsilon D + (1/\varepsilon)A$, for $\varepsilon > 0$, we can write

(4) $\quad D \le 2D^{1/2}A^{1/2} + 2\operatorname{sgn}(\Delta g)B \le \varepsilon D + (1/\varepsilon)A + 2\operatorname{sgn}(\Delta g)B, \quad$ for $\varepsilon > 0$.

Now, for any $\delta \ge 0$,

$$\delta C - \delta D/4 = \delta[4C - D]/4 \ge 0,$$

since $4C \ge D$ by Lemma 3. Combining this result with (4) we have

$$D \le \varepsilon D + A/\varepsilon + 2\operatorname{sgn}(\Delta g)B + \delta C - \delta D/4,$$

for all $\varepsilon > 0$ and $\delta \geq 0$. Rewriting we obtain

$$A/\varepsilon \geq D - \varepsilon D - 2\,\mathrm{sgn}(\Delta g)B - \delta C + \delta D/4,$$

thus,

$$A \geq -[2\varepsilon\,\mathrm{sgn}(\Delta g)B + \varepsilon\delta C] + \varepsilon(1 - \varepsilon + \delta/4)D,$$

and the proof is complete. □

Corollary 6. *Suppose* $n \geq 2$ *and* $\alpha \in [(-\infty, -2) \cup (n-4, +\infty)]$, $\alpha \neq -n$. *Then if* $s \in [\frac{1}{2}(\alpha+n)(4+\alpha-n), \infty)$ *and* $u \in C_0^\infty(\mathbf{R}_+^n)$, *we have that*

$$\int_{\mathbf{R}_+^n} |x|^{\alpha+4}|\Delta u|^2\,dx \geq -s\int_{\mathbf{R}_+^n} |x|^{\alpha+2}|\nabla u|^2\,dx + \frac{(\alpha+n)^2[(4+\alpha-n)^2+4s]}{16} \cdot \int_{\mathbf{R}_+^n} |x|^\alpha |u|^2\,dx. \tag{5}$$

Proof. Let $g = |x|^{\alpha+2}$; α real; $\alpha \neq -2, -n$ in inequality (2). Then $|\nabla g|^2 = (\alpha+2)^2|x|^{2\alpha+2}$, and $\Delta g = (\alpha+n)(\alpha+2)|x|^\alpha$. Inequality (2) becomes

$$\begin{aligned} \int_{\mathbf{R}_+^n} |x|^{\alpha+4}|\Delta u|^2\,dx &\geq -\varepsilon(\alpha+2)[2(\alpha+n)+\delta(\alpha+2)]\int_{\mathbf{R}_+^n} |x|^{\alpha+2}|\nabla u|^2\,dx \\ &\quad + \varepsilon(\alpha+n)^2(\alpha+2)^2\left(1-\varepsilon+\frac{\delta}{4}\right)\int_{\mathbf{R}_+^n} |x|^\alpha|u|^2\,dx. \end{aligned} \tag{6}$$

Let $s = \varepsilon(\alpha+2)[2(\alpha+n)+\delta(\alpha+2)]$. Rewriting (5) we have

$$\begin{aligned} \int_{\mathbf{R}_+^n} |x|^{\alpha+4}|\Delta u|^2\,dx &\geq s\left[-\int_{\mathbf{R}_+^n} |x|^{\alpha+2}|\nabla u|^2\,dx + \frac{(\alpha+n)^2}{4}\int_{\mathbf{R}_+^n} |x|^\alpha|u|^2\,dx\right] \\ &\quad + \frac{(\alpha+n)^2(4+\alpha-n)^2}{16}\int_{\mathbf{R}_+^n} |x|^\alpha|u|^2\,dx. \end{aligned} \tag{7}$$

Letting $\beta = \alpha + 2$ in Corollary 4, we obtain that

$$\int_{\mathbf{R}_+^n} |x|^{\alpha+2}|\nabla u|^2\,dx \geq \frac{(\alpha+n)^2}{4}\int_{\mathbf{R}_+^n} |x|^\alpha|u|^2\,dx,$$

and hence

$$-\int_{\mathbf{R}_+^n} |x|^{\alpha+2}|\nabla u|^2\,dx + \frac{(\alpha+n)^2}{4}\int_{\mathbf{R}_+^n} |x|^\alpha|u|^2\,dx \leq 0.$$

Thus, $s = (\alpha+n)(4+\alpha-n)/2$ is the best constant for (7). To establish (7) for this s we choose $\delta = 0$ in $s = \varepsilon(\alpha+2)[2(\alpha+n)+\delta(\alpha+2)]$, and consequently we have $\varepsilon = (4+\alpha-n)/4(\alpha+2)$ which is positive by hypothesis. With this choice of δ, ε, and s, after a calculation we get

$$\varepsilon(\alpha+n)^2(\alpha+2)^2\left(1-\varepsilon+\frac{\delta}{4}\right)=\frac{(\alpha+n)^2[(4+\alpha-n)^2+4s]}{16}.$$

Thus (6) implies (7) for $s = \frac{1}{2}(\alpha+n)(4+\alpha-n)$ and hence for all $s \in [\frac{1}{2}(\alpha+n)(4+\alpha-n), \infty)$. □

By choosing $\alpha = -4$ in Corollary 6 we have the result of Lemma 2 (Schmincke [5]).

Allegretto's result is improved for $\alpha \in (-n, -2)$ when $n \geq 3$ by taking $s = 0$ in Corollary 6.

Lemma 3 (Lewis [3]) indicates that it should be possible to extend Theorem 5 to functions other than $\theta \in C_0^\infty(\Omega)$.

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