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TERMS IN THE SELBERG TRACE FORMULA FOR $SL(3,\mathbf{Z})\backslash SL(3,\mathbf{R})/SO(3,\mathbf{R})$ ASSOCIATED TO EISENSTEIN SERIES COMING FROM A MAXIMAL PARABOLIC SUBGROUP

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ABSTRACT. There are two types of Eisenstein series associated to $SL(3,\mathbf{Z})$. This paper deals with those which are built out of cuspidal Maass waveforms for $SL(2,\mathbf{Z})$. We compute the inner product of two of them over a truncated fundamental region and then compute the rate of divergence as the truncation parameter tends to infinity. The solution of this problem is of use in computing the details of the trace formula for $SL(3,\mathbf{Z})$.

The Selberg trace formula for a discrete group Γ acting on $SL(3,\mathbf{R})/SO(3,\mathbf{R})$ is of interest to number theorists because it is one of the first examples of rank > 1 to be worked out in detail. Arthur [1] has obtained results for $SL(n,\mathbf{R})$ showing that relevant operators are indeed trace class. His methods do not provide a version which allows one to obtain truly analytical results, however. A detailed computation of the contribution to the trace formula of orbital integrals and Eisenstein series would enable one, for example, to compute the rate of growth of eigenvalues associated to the discrete specrum of the Laplace–Beltrami operator, as in Hejhal [2]. A more ambitious application would be computing the rate of growth of class numbers of cubic number fields as compared with the size of their regulators, as in Sarnak [5]. Both of these applications arise primarily when $\Gamma = SL(3,\mathbf{Z})$, which is the case considered in this paper.

The principal difficulty in computing with the trace formula arises out of the combined contribution of the various types of Eisenstein series and the orbital integrals over elements in various parabolic subgroups. Both types of objects are badly divergent in their own right, but together they contribute a finite amount to the formula. The general shape of the formula is

$$(1)\quad \operatorname{Tr} k_f = \int_{\mathcal{F}} \left[\sum_{\gamma\in\Gamma} f(z^{-1}\gamma z) - \sum_{i,\phi} \int_\lambda h(\phi,\lambda) E(i,\phi,\lambda)\overline{E(i,\phi,\lambda)}\, d\lambda \right] dz,$$

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where the indices (i, ϕ, λ) are supposed to count the various parts of the continuous spectrum. $\mathscr{F}$ is a fundamental region for the action of Γ. In order to compute anything one must be able to separate the various parts of the right-hand side of (1), which can only be done if a variable of truncation is introduced. That is, the fundamental region is cut off in such a way that the integrals converge separately. They are then computed and later the cutoff is pushed towards the cusp.

All of the orbital integrals which diverge badly have been computed in this manner in Wallace [9, 10]. An effort is made in this paper to match the truncation parameter with the one used in these papers so that we can later compare these terms and compute the contribution of their total. The situation looks quite promising because in almost every case the orbital integrals can be reduced to orbital integrals occurring in the Selberg trace formula for a rank 1 reductive group. These observations justify the calculations in this paper in that we only attempt to reduce the contribution of a particular Eisenstein series to a rank 1 contribution associated to the same trace formula as in the above-mentioned integrals.

The spectrum of the Laplace–Beltrami operator on $\mathrm{SL}(3,\mathbf{Z})\backslash \mathrm{SL}(3,\mathbf{R})/\mathrm{SO}(3,\mathbf{R})$ breaks into several pieces. The one treated in this paper consists of Eisenstein series built out of cusp forms on $\mathrm{SL}(3,\mathbf{R})/\mathrm{SO}(2,\mathbf{R})$. Fix the parabolic subgroup of $\mathrm{SL}(3,\mathbf{R})$ which we will call P_1 consisting of elements of the form

$$\left(\begin{array}{cc|c} \multicolumn{2}{c|}{*} & * \\ \hline 0 & 0 & * \end{array}\right)$$

P_1 has a decomposition, called the Langlands decomposition, which is given by the product below. We abbreviate $\mathrm{SO}(3,\mathbf{R})$ by the letter K.

$P_1 = N_1 A_1 M_1$ where

$$n_1 = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad a_1 = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1^{-2} \end{pmatrix}, \qquad m_1 = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\tilde{m}_1} & 0 \\ & & 0 \\ \hline 0 & 0 & \pm 1 \end{array}\right)$$

where $\tilde{m}_1 \in \mathrm{GL}(2,\mathbf{R})$. Further, m_1 can be written

$$m_1 = \begin{pmatrix} u_1^{1/2} & v_1 u_1^{-1/2} & 0 \\ 0 & u_1^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot k_1 = z_1 k_1,$$

where $k_1 \in \mathrm{SO}(3,\mathbf{R}) \cap P_1$ and $v_1 + iu_1$ is a coset representative of $\mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2,\mathbf{R})$ and hence can be regarded as an element of the Poincaré upper half plane, $\{v_1 + iu_1, u, v_1 \in \mathbf{R}, u_1 > 0\}$. We can build an Eisenstein series out of an element of the discrete spectrum for $\mathrm{SL}(2,\mathbf{Z})\backslash \mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2,\mathbf{R})$ using this decomposition. Let $v(z)$ be a cusp form for $\tilde{\Gamma} = \mathrm{SL}(3,\mathbf{Z}) \cap M_1$ on the symmetric space $M_1/M_1 \cap K$, normalized to have norm 1 with respect to

$L^2(\widetilde{\Gamma}\backslash M_1/M_1 \cap K)$. Set

$$E_1(Y;v,s) = \sum_{\gamma\in P_1\backslash\Gamma} y_1(\gamma Y)^s v(z_1(\gamma Y))$$

where $Y = n_1 a_1 z_1 K$ and $y_1 = \alpha_1^6$ and $z_1 = v_1 + iu_1$. Haar measure for this decomposition is given by

$$dY = c_0 \frac{dy_1\, dz_1\, dx_1\, dt_1}{y_1^2 u_1^2},$$

where $c_0 = \frac{1}{6}$ in order to match previous work.

Here P_1 is an example of a maximal parabolic subgroup of SL$(3,\mathbf{R})$. Up to conjugacy there is one more, P_2, consisting of elements of the form

$$\left(\begin{array}{c|c} * & * \\ \hline 0 & * \\ 0 & \end{array}\right)$$

and it comes equipped with its Langlands decomposition

$$\begin{aligned} P_2 &= N_2 A_2 M_2 \\ &= \begin{pmatrix} 1 & x_2 & t_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_2^{+2} & & \\ & \alpha_2^{-1} & \\ & & \alpha_2^{-1} \end{pmatrix} \begin{pmatrix} \pm 1 & 0 \quad 0 \\ 0 & \left[\; m_2 \;\right] \\ 0 & \end{pmatrix} \end{aligned}$$

so that a second collection of Eisenstein series can be constructed. Set

$$E_2(Y;v,s) = \sum_{\gamma\in P_2\backslash\Gamma} y_2(\gamma Y)^s v(z_2(\gamma Y)).$$

Both types of Eisenstein series are described along with their relation to the rest of the spectrum in Venkov [8] and it is his spectral decomposition we draw on here. These two types of Eisenstein series obey the functional equations

$$\begin{aligned} E_1(Y;c_2(v,s), 1-s) &= E_2(Y;v,s), \\ E_2(Y;c_1(v,s), 1-s) &= E_1(Y;v,s), \end{aligned}$$

where $c_j(v,s)$ is another cusp form with the same eigenvalue as v and

$$c_1(c_2(v,s), 1-s) = c_2(c_1(v,s), 1-s) = v.$$

Although the two types of Eisenstein series are not orthogonal, due to the functional equations they contribute the same amount to the spectrum. Hence we only calculate the contribution for P_1. Also, because of the functional equations, we have that the Fourier expansion of E_1 has the constant term

$$y_1^s v(z_1) + y_2^{1-s} c_2(v,s)(z_2) \tag{2}$$

(see Langlands [4]). The poles of these in the region $\operatorname{Re} s \geq \frac{1}{2}$ are located on the real axis in the interval $(\frac{1}{2}, 1]$ and do not depend on Y or on the choice of v in a given eigenspace. The contribution to the trace formula of functions of these types is

$$(3)\qquad \frac{-1}{2\pi i}\sum_{v_i}\int_{\mathcal{F}_A}\int_{\operatorname{Re}s=\frac{1}{2}} h(s,v_i)E_1(Y,v_i,s)\overline{E_1(Y,v_i,s)}\,ds\,dY,$$

where $h(s,v_i)$ is the Selberg transform of $f(Y)$, dY is Haar measure on $SL(3,\mathbf{R})/SO(3,\mathbf{R})$, the v_i constitute an orthonormal basis for the point spectrum in the Poincaré upper half plane, and $\mathcal{F}_A$ is a choice of fundamental region for Γ with its cusp at $y=\infty$, truncated in such a way that $y<A^6$. This truncation matches those in Wallace [9, 10].

In addition to these functions more parts of the spectral decomposition arise when one takes the residue of an Eisenstein series at one of its poles in $(\frac{1}{2},1]$. This gives more possible spectrum of the form:

$$(4)\qquad \frac{-1}{2\pi i}\sum_{v_i}\sum_{s_{ij}}\int_{\mathcal{F}_A} h(s_{ij},v_i)\operatorname{Res}_{s_{ij}}(E(Y;v_i,s))\overline{\operatorname{Res}_{s_{ij}}(E(Y;v_1,s))}\,dY.$$

It is possible to include the contribution of (4) in the discrete spectrum. We now wish to move the line of integration in (3) over to a place where the Eisenstein series converges, namely, $\operatorname{Re}s = 1+\varepsilon$. The space spanned is still orthogonal to the various other parts of the spectrum described in Venkov [8], but the eigenvalues used are no longer the L^2 eigenvalues for the Laplacian. When our computation is finished we must move the line of integration back to $\operatorname{Re}s=\frac{1}{2}$. In order for the move to make sense, we must replace $\bar{s}$ by $1-s$ in (3). Then we must worry about poles of the product $E(Y,\nu,s)E(Y,\bar{\nu},1-s)$. Since $c(\nu,s)$ is a unitary operator, it does not have poles. Therefore any residue of this product will not have a constant term in its Fourier expansion. Thus we can do the integral over $\mathcal{F}$ and we will just get zero as its contribution to the trace formula.

We now set out to compute

$$(5)\ c_0\frac{-1}{2\pi i}\sum_{v_i}\int_{\mathcal{F}_A}\int_{\operatorname{Re}s=1+\varepsilon} h(s,v_i)E_1(Y,v_i,s)E_1(Y,\bar{v}_i,1-s)\,ds\,dY$$
$$= c_0\frac{-1}{2\pi i}\sum_{v_i}\int_{\operatorname{Re}s=1+\varepsilon} h(s,v_i)\left[\int_{\mathcal{F}_A} E_1(Y,v_i,s)E_1(Y,\bar{v}_i,1-s)\,dY\right]ds$$

which in turn leads us to compute

$$(6)\qquad \int_{\mathcal{F}_A} E_1(Y,v_i,s)E_1(Y,\bar{v}_i,1-s)\,dY.$$

Now, we are in a region of convergence for the Eisenstein series, so we can unwind it and (6) becomes

$$(7)\qquad \sum_{\gamma\in P_1\backslash\Gamma}\int_{\gamma\circ\mathcal{F}_A} v_i(z_1)y_1^s E_1(Y,\bar{v}_i,1-s)\,dY.$$

At this point in the calculations we wish to replace the regions of integration

$$\sum_{\gamma \in P_1 \backslash \Gamma} \int_{\gamma \circ \mathscr{F}_A} \tag{8}$$

with the region

$$\frac{1}{4} \int_{y_1 = 1/A^6}^{A^6} \int_{z_1 \in \widetilde{\mathscr{F}}} \int_{x_1 = 0}^{1} \int_{t=0}^{1}, \tag{9}$$

where $\widetilde{\mathscr{F}}$ is a fundamental region for $\mathrm{SL}(2, \mathbf{Z})$ in the upper half plane. First notice that as A approaches ∞, both of these regions approach the same region. That is, the strip,

$$S = \{Y | 0 < y_1 < \infty,\ z_1 \in \widetilde{\mathscr{F}},\ 0 < x_1 \le 1,\ 0 < t_1 \lesssim 1\}$$

contains four copies of a fundamental region for P_1. In order to justify this change of truncation parameter we must estimate the difference,

$$\left| \sum_{\gamma \in P_1 \backslash \Gamma} \int_{\gamma \circ \mathscr{F}_A} v_i(z_1) y_1^s E_1(Y, \bar{v}_i, 1 - s)\, dY - \frac{1}{4} \int_{y_1 = 1/A^6}^{A^6} \int_{z_1 \in \widetilde{\mathscr{F}}} \int_{x_1 = 0}^{1} \int_{t=0}^{1} v_i(z_1) y_1^s E_1(Y, \bar{v}_i, 1 - s)\, dY \right| \tag{10}$$

as A tends to ∞. Now, (9) can be written as

$$\int_{\mathscr{F}_A} \left[\sum_{\substack{\gamma \in P_1 \backslash \Gamma \\ y_1(\gamma Y) > 1/A^6}} y_1(\gamma Y)^s v_i(z_1(\gamma Y) \right] E_1(Y, \bar{v}_i, 1 - s)\, dY \tag{11}$$

and (8) can be written as before, in (6). Putting these together gives that (10) is equal to

$$\left| \int_{\mathscr{F}_A} \left[E_1(Y, v_i, s) - \sum_{\substack{\gamma \in P_1 \backslash \Gamma \\ y_1(\gamma Y) > 1/A^6}} y_1(\gamma Y)^s v_i(z_1(\gamma Y)) \right] E_1(Y, \bar{v}_i, 1 - s)\, dY \right| \tag{12}$$

which equals

$$\left| \int_{\mathscr{F}_A} \left(\sum_{\substack{\gamma \in P_1 \backslash \Gamma \\ y_1(\gamma Y) \le 1/A^6}} y_1(\gamma Y)^s v_i(z_1(\gamma Y)) \right) E_1(Y, \bar{v}_i, 1 - s)\, dY \right| \tag{13}$$

$$\le \int_{\mathscr{F}_A} \left| \sum_{\substack{\gamma \in P_1 \backslash \Gamma \\ y_1(\gamma Y) \le 1/A^6}} y_1(\gamma Y)^s v_i(z_1(\gamma Y)) \right| |E_1(Y, \bar{v}_i, 1 - s)|\, dY \tag{14}$$

which leads us to estimate the quantity

$$(15)\qquad \left| \sum_{\substack{\gamma\in P_1\backslash\Gamma \\ y_1(\gamma Y)\le 1/A^6}} y_1(\gamma Y)^s v_i(z_1(\gamma Y)) \right| .$$

The highest height for any $y(\gamma Y)$ in the sum, (15), is less than A^{-6} and so (15) is roughly

$$O(A^{-6\varepsilon})\cdot N(A^{-6})$$

where $N(A^{-6})$ is the number of y's at height A^{-6}, which is less than $\ln A$. So (15) is approximately $O(A^{-6}\ln A)$. On the other hand, the integral

$$(16)\qquad \int_{\mathcal{F}_A} |E_1(Y;\bar{v}_i, 1-s)|\,dY$$

can also be estimated because $E_1(Y;\bar{v}_i,1-s)$ is $O(y_1^s)$ as y_1 gets large. So the integral (16) is roughly

$$\int_{\mathcal{F}_A} |y_1^{1+\varepsilon+it}|\frac{dy}{y_1^2}\frac{du_1\,dv_1}{u_1^2}\,dx_1\,dt_1 = c\int_{y_1=1}^{A^6} y_1^{1+\varepsilon}\frac{dy_1}{y_1^2} = c\int_{y_1=1}^{A^6}\frac{dy_1}{y_1^{1-\varepsilon}}$$

which approaches $(6\ln A)$ as $\varepsilon\to 0$. So the whole of (15) is roughly

$$O(A^{-6\varepsilon}(\ln A)^2)$$

which approaches zero as A gets large. Thus we are justified in replacing the integral given by (7) with

$$(17)\qquad \frac{1}{4}\int_{y_1=1/A^6}^{A^6}\int_{z_1\in\tilde{\mathcal{F}}}\int_{x_1=0}^{1}\int_{t_1=0}^{1} y_1^s v_i(z_1)E_1(Y,\bar{v}_i,1-s)\,dY .$$

Now, the integral

$$(18)\qquad \int_{t_1=0}^{1} E_1(Y,\bar{v}_i,1-s)\,dx_1\,dt_1$$

yields the constant term in the Fourier expansion for this Eisenstein series (given in (2)) which is

$$y_1^{1-s}\bar{v}_i(z_1)+y_2^{1-s}c_2(\bar{v}_i,1-s)(z_2),$$

so (17) becomes

$$(19)\qquad \begin{aligned} &\frac{1}{4}\int_{y_1=A^{-6}}^{A^6}\int_{z_1\in\tilde{\mathcal{F}}}\int_{x_1=0}^{1} y_1\bar{v}_i(z_1)v_i(z_1)\frac{dy_1}{y_1^2}\frac{dz_1}{u_1^2}\,dx_1 \\ &+\frac{1}{4}\int_{y_1=A^{-6}}^{A^6}\int_{z_1\in\tilde{\mathcal{F}}}\int_{x_1=0}^{1}(y_1^s v_i(z_1))(y_2^{1-s}c_2(\bar{v}_i,1-s)(z_2))\frac{dy_1\,dz_1\,dx_1}{y_1^2u_1^2} .\end{aligned}$$

The second term in the sum, (19), contains the integral

$$\int_{x_1=0}^{1} c_2(\bar{v}_i,1-s)(z_2)\,dx_1$$

which is the integral over a closed horocycle in the z_2-plane and hence zero because c_2 is a cusp form. The first term in (19) contains the integral

$$\int_{z_1 \in \tilde{\mathscr{F}}} |v_i(z_1)|^2 \frac{dz_1}{u_1^2}$$

which is 2 because the v_i are normalized in exactly this way. So (19) becomes

$$\frac{1}{2} \int_{y_1 = A^{-6}}^{A^6} \frac{dy_1}{y_1} = 6 \ln A .$$

Therefore the total contribution of (5) is

$$c_0 \frac{-1}{2\pi i} \cdot 6 \ln A \sum_i \int_{\operatorname{Re} s = 1+\varepsilon} h(s, v_i)\, ds . \tag{20}$$

Now we move the line of integration to $\operatorname{Re} s = \frac{1}{2}$, which can be done if $h(s, v_i) \to 0$ as $|\operatorname{Im} s| \to \infty$, to give

$$-6 \ln A \cdot c_0 \sum_{v_i} \frac{1}{2\pi i} \int_{\operatorname{Re} s = 1/2} h(s, v_i)\, ds = -\ln A \sum_{\lambda_i} \mathscr{h}(\lambda_i) \tag{21}$$

where $\mathscr{h}(\lambda_i) = (1/2\pi i) \int_{\operatorname{Re} s = 1/2} h(s, v_i)\, ds$ and λ_i is the eigenvalue associated to v_i in the Poincaré upper half plane, counted with multiplicity. It remains to interpret the transform $\mathscr{h}(\lambda_i)$ as the Selberg transform of some suitable kernel $\tilde{f}$ on some lower rank subgroup of $\mathrm{SL}(3, \mathbf{R})$.

Let f be the K-bi-invariant kernel under discussion in this paper. We define the function

$$\hat{f}(a_1 m_1) = \int_{n_1} f(a_1 m_1 n_1)\, dn_1 \tag{22}$$

where a_1, m_1, n_1 are defined as before to be the various parts of the Langlands decomposition for P_1. We call this the Harish–Chandra transform of the function f relative to the parabolic subgroup P_1. (If P were the minimal parabolic subgroup this would be what is called the Harish transform in Lang [3] and Terras [7]. If G were $\mathrm{SL}(2, \mathbf{R})$ this transform would be equivalent to the Abel transform used in Selberg [6] and Hejhal [2].)

Notice that $\hat{f}$ is no longer K-bi-invariant. However it is still right K-invariant since f is a function on the symmetric space. On the left it is invariant under the compact group $M_1 \cap K$, which is just matrices of the form

$$m_0 \times \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathrm{SO}(2, \mathbf{R})} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array} \right)$$

where $m_0 \in M_0$, the subgroup of diagonal matrices with ± 1 on the diagonal. M_0 has order four. Hence we interpret $\hat{f}$ as a $K \cap M_1$-bi-invariant kernel on the symmetric space $M_1 / K \cap M_1$, for a fixed $a_1 \in A_1$.

The Selberg transform associated to a kernel f is defined by the relation

$$\int_{\mathcal{F}} F(Y^{-1}Z)E(Z,\phi,s)\,dZ = h(s,\phi)E(Y,\phi,s) \tag{23}$$

where $F(Y^{-1}Z) = \sum_{\gamma\in\Gamma} f(Y^{-1}\gamma Z)$. Note that the left-hand side in (23) is equal to

$$\int_{Z\in \mathrm{SL}(3,\mathbf{R})/\mathrm{SO}(3,\mathbf{R})} f(Y^{-1}Z)E(Z,\phi,s)\,dZ. \tag{24}$$

It is a lemma due to Selberg that the value of $h(s,\phi)$ depends only on the eigenvalues associated to the eigenfunction against which f is integrated. So we replace relation (23) by

$$\int_{Z\in \mathrm{SL}(3,\mathbf{R})/\mathrm{SO}(3,\mathbf{R})} f(Y^{-1}Z)(y_1(Z)^s u_1(Z)^t)\,dZ = h(s,\phi)y_1(Y)^s u_1(Y)^t \tag{25}$$

for some t. Letting $Y =$ identity, we obtain

$$h(s,\phi) = \int_{Z\in \mathrm{SL}(3,\mathbf{R})/\mathrm{SO}(3,\mathbf{R})} f(Z)(y_1(Z)^s u_1(Z)^t)\,dZ.$$

We now compute

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\mathrm{Re}\,s=1/2} h(s,\phi)\,ds\\ &\quad= \frac{1}{2\pi i}\int_{\mathrm{Re}\,s=1/2}\int_{\gamma\in \mathrm{SL}(3,\mathbf{R})/\mathrm{SO}(3,\mathbf{R})} f(Y)y(Y)^s u(Y)^t\,dY\,dS.\end{aligned} \tag{26}$$

Thinking of f as a function of the five-tuple $(y_1, u_1, v_1, x_1, t_1)$ and recalling that

$$dY = \frac{du_1}{u_1^2}\,dv_1\,\frac{dy_1}{y_1^2}\,dx_1\,dt_1$$

we notice that part of the integral in (26) is given by the expression

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\,s=1/2}\int_{y_1=0}^{\infty} f(y_1, u_1, v_1, x_1, t_1)y_1^s\frac{dy_1}{y_1^2}\,dS,$$

which is just the Mellin transform of f in the variable y_1, followed by Mellin inversion at $y_1 = 1$ (Terras [7]). Hence (26) becomes

$$\begin{aligned}&\int_{u_1=0}^{\infty}\int_{v_1=-\infty}^{\infty}\int_{x_1=-\infty}^{\infty}\int_{t_1=-\infty}^{\infty} f(1, u_1, v_1, x_1, t_1)u_1^t\frac{du_1\,dv_1}{u_1^2}\,dx_1\,dt_1\\ &\quad= \int_{u_1=0}^{\infty}\int_{v_1=-\infty}^{\infty} \hat{f}(1,u,v)u_1^t\frac{du_1\,dv_1}{u_1^2}.\end{aligned} \tag{27}$$

Set $\tilde{f}(u,v) = \hat{f}(1,u,v)$. Then (27) is just the Selberg transform of $\tilde{f}$ on $M_1/M_1\cap K$.

We now summarize the results of this paper in the following theorem.

Theorem. *The total contribution to the trace formula for* $\mathrm{SL}(3,\mathbf{Z})$ *of all Eisenstein series associated to both maximal parabolic subgroups which are induced from the discrete cuspidal spectrum of* $\mathrm{SL}(2,\mathbf{Z})$ *is given by the following amount*:

$$-2\ln A \sum_{\lambda i} \hbar(\lambda i)$$

where $\hbar(\lambda i)$ *is the Selberg transform of the function* $f(v+iu)$ *described above.*

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