

RADÓ'S THEOREM FOR CR-FUNCTIONS

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(Communicated by Irwin Kra)

ABSTRACT. We establish an analogue of a classical theorem of Radó for CR-functions on certain hypersurfaces in \mathbb{C}^N .

I. INTRODUCTION

A well-known theorem of Radó states that a continuous function defined on an open set in \mathbb{C} that is holomorphic on the complement of its zero set is holomorphic everywhere. The result is correct in \mathbb{C}^N as well as in the plane. The usual proofs rely on properties of subharmonic functions [11, 14] or on methods of the theory of uniform algebras [3, 19]. Our purpose here is to obtain an analogue of Radó's theorem in case of CR-functions defined on hypersurfaces in \mathbb{C}^N . Our analysis does not cover the case of completely general hypersurfaces; for our proof we need to impose conditions on the size of the set where the Levi form vanishes.

We obtain two results, the first of which is as follows.

Proposition 1. *If Σ is a strictly pseudoconvex hypersurface of class \mathcal{E}^2 in an open set in \mathbb{C}^2 , and if u is a continuous function on Σ the restriction of which to $\Sigma \setminus u^{-1}(0)$ is a CR-function, then u is a CR-function on Σ .*

It then follows that u extends locally to the pseudoconvex side of Σ as a holomorphic function.

As usual, a CR-function on Σ is defined to be a function that satisfies the tangential Cauchy-Riemann equations $\bar{\partial}_b u = 0$, which, in case that u merely continuous, means that

$$\int_{\Sigma} u \bar{\partial}_b \varphi \wedge dz_1 \wedge dz_2 = 0$$

for all $\varphi \in \mathcal{E}^1(\Sigma)$ with compact support in Σ .

Received by the editors February 22, 1988 and, in revised form, October 14, 1988.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 32A40, 32F25.

The first author's research was supported in part by Grant 8800610 from the National Science Foundation.

The second author's research was supported in part by Grants 8601131 and 8801032 from the National Science Foundation.

The proof we give for Proposition 1 is local, so it follows immediately that in the statement, we can replace the set $u^{-1}(0)$ by $u^{-1}(E)$ for any countable compact set E . Herbert Alexander has shown recently that, as in the classical case [18], it is possible to replace $u^{-1}(0)$ by $u^{-1}(E)$, E a compact set of zero capacity.

To formulate our second result, we need a definition.

Definition. If Σ is a hypersurface in an open set in C^N , a closed subset E of Σ will be said to be *thin* if every closed subset E' of E has a neighborhood basis $\{V_j\}_{j=1,2,\dots}$ each element of which has piecewise smooth boundary bV_j in Σ and these boundaries have $(2N-2)$ -dimensional area bounded uniformly in j .

Examples of such thin sets are sets of finite $(2N-2)$ -dimensional measure.

Proposition 2. *Let Σ be a hypersurface of class \mathcal{E}^2 in an open subset of C^N , $N > 1$, and assume that the set of points in Σ where the Levi form of Σ vanishes, i.e., all eigenvalues are zero, is thin. Then every continuous function u on Σ the restriction of which to $\Sigma \setminus u^{-1}(0)$ is a CR-function is a CR-function on all of Σ .*

Our consideration of these matters can be motivated in at least two ways. The first motivation comes from the study of *removable sets* in the boundary of a domain Ω in C^N : subsets $F \subset b\Omega$ such that every CR-function defined on $b\Omega \setminus F$ extends holomorphically through the set Ω . For this work, we refer to the papers [7, 8, 9, 20].

Another motivation comes from a general problem about complex vector fields. Consider on an open subset U in R^N a complex vector field $L = \sum_{j=1}^N a_j(x) \partial / \partial x_j$. Let f be a continuous function on U that satisfies $Lf = 0$ in the sense of distributions on $U \setminus f^{-1}(0)$. Does it follow that $Lf = 0$ throughout U ? When $N = 2$ and L is the Cauchy-Riemann operator, the result is the classical theorem of Radó quoted above, when $L = \partial / \partial x_1$, the result is true and trivial, and the case that L is the tangential Cauchy-Riemann operator on a hypersurface in C^N is the object of this paper. We have no example of a vector field for which the conclusion fails, and the paper [6] may suggest that the result is true in general. It seems unlikely that the proofs we give below shed any light on the general question, for we reduce the problem for CR-functions to the Radó's theorem on C^N .

Our work has the following consequence:

Corollary. *If D is a strongly pseudoconvex domain in C^N with bD of class \mathcal{E}^2 , and if the function u is continuous on \bar{D} , holomorphic on D , then $bD \setminus u^{-1}(0)$ is connected.*

The paper concludes with an application of Proposition 1 to the study of removable singularities for bounded solutions of $\bar{\partial}_b$.

We shall use the notation that if K is a compact set in \mathbb{C}^N , then \widehat{K} denotes its polynomially convex hull. If $A \subset \mathbb{C}^N$, bA will denote the boundary of A , boundary with respect to \mathbb{C}^N unless otherwise specified.

II. PROOF OF PROPOSITION 1

We begin with a lemma which is a special case of a result of Słodkowski [15, Theorem 2.1, p. 365]; for the convenience of the reader, we include a proof.

Lemma. *If $K \subset \mathbb{C}^2$ is a compact set and $\mathcal{O} \subset \mathbb{C}^2$ is pseudoconvex, then provided that $K \cap \mathcal{O}$ is empty, the set $\mathcal{O} \setminus \widehat{K}$ is pseudoconvex.*

Proof. Assume, in order to reach a contradiction, that $\mathcal{O} \setminus \widehat{K}$ is not pseudoconvex. Denote by U the open unit disc in \mathbb{C} . According to the *Kontinuitätssatz* [21], there exists a change of coordinates in \mathbb{C}^2 such that after this change of coordinates, the unit bidisc U^2 does not intersect K , and there exists $\{\varphi_j\}_{j=1}^\infty$, a sequence of holomorphic functions on \mathbb{C} that satisfy $|\varphi_j| < \frac{1}{2}$ on U and such that, if we set $\Phi_j(s) = (s, \varphi_j(s))$, then Φ_j maps \overline{U} into $\mathcal{O} \setminus \widehat{K}$, such that $\text{dist}(\Phi_j(e^{i\theta}), \widehat{K}) > \delta$, and such that $\Phi_j(0)$ tends to some point $p_0 \in \widehat{K}$ as k tends to infinity.

If we set $f_j(z_1, z_2) = 1/(z_2 - \varphi_j(z_1))$, then f_j is a holomorphic function defined on a neighborhood of $\widehat{K} \cap \overline{U}^2$. By the Oka-Weil theorem, f_j can be approximated uniformly on the set $\widehat{K} \cap \overline{U}^2$ by polynomials. Since we have $|f_j| \leq \max(\delta^{-1}, 2)$ on $\widehat{K} \cap bU^2$ but the supremum of $|f_j|$ on $\widehat{K} \cap U^2$ tends to infinity with j , we have a contradiction to the local maximum principle [13, 17, 19], so the proof is completed.

We also need the following more technical fact.

Lemma. *Let \mathcal{O} be a bounded pseudoconvex domain in \mathbb{C}^2 , and let K be a compact subset of $b\mathcal{O}$ with the property that $\widehat{K} \cap b\mathcal{O} = K$. If $\mathcal{O}' = \mathcal{O} \setminus \widehat{K}$, and if ω is a neighborhood of K , then there exists an open set $\Omega \subset \mathcal{O}'$ that contains the intersection of \mathcal{O}' with some neighborhood of \widehat{K} and that satisfies the condition that for every function h holomorphic on \mathcal{O}' ,*

$$\sup_{\Omega} |h| = \sup_{\mathcal{O}' \cap \omega} |h|.$$

Proof. We consider small open neighborhoods W_1 of \widehat{K} and W_2 of $\overline{b\mathcal{O} \setminus \widehat{K}}$ such that $\overline{W_1} \cap \overline{W_2} \subset \omega$. The domain \mathcal{O} is pseudoconvex, so by the lemma above, there exists a smoothly bounded, strictly pseudoconvex domain $\mathcal{O}_1 \subset \subset \mathcal{O}'$ such that $\mathcal{O}_1 \supset \mathcal{O}' \setminus (\overline{W_1} \cup \overline{W_2})$. Let $\Omega = (\mathcal{O}' \setminus \overline{\mathcal{O}_1}) \cap W_1$. The domain Ω is the intersection of \mathcal{O}' with a neighborhood of \widehat{K} , and the boundary of Ω is included in $\widehat{K} \cup \omega \cup \mathcal{O}_1$. At each point of $b\Omega \setminus (\widehat{K} \cup \omega)$, the boundary of Ω is strictly pseudoconcave when considered from Ω . We shall show that if h is holomorphic on \mathcal{O} and $p \in \Omega$, then $|h(p)| < \sup_{\omega \cap \mathcal{O}'} |h|$. To this end, consider W_3 , a small strictly pseudoconvex neighborhood of K such that

$p \notin \overline{W}_3$, $\overline{W}_3 \cap \overline{\mathcal{O}}_1 = \emptyset$. Let $\Omega_p = \Omega \setminus \overline{W}_3$. The boundary of Ω_p consists of points that lie in ω or at which the boundary of Ω_p is strictly pseudoconcave: The part of the boundary of Ω_p outside ω coincides locally with $b\mathcal{O}_1$ or with bW_3 . Since no local maximum can be attained a strictly pseudoconcave point, we get that for every function h holomorphic on \mathcal{O}'

$$|h(p)| \leq \sup_{\Omega_p} |h| \leq \sup_{\Omega_p \cap \omega} |h| \leq \sup_{\mathcal{O}' \cap \omega} |h|.$$

We turn now to the proof of Proposition 1. The problem is purely local, so as strongly pseudoconvex hypersurfaces are strictly convex in suitable local coordinates, it suffices to establish the following version of the result.

Proposition 1'. *Let D be a bounded, strictly convex domain in \mathbb{C}^2 with bD of class \mathcal{E}^2 , and suppose that $0 \in D$. Let $\mathcal{O} = D \cap \{\operatorname{Re} z_1 > 0\}$, and let $(b\mathcal{O})^+ = bD \cap \{\operatorname{Re} z_1 > 0\}$. If u is a bounded continuous function on $(b\mathcal{O})^+$ the restriction of which to $(b\mathcal{O})^+ \setminus u^{-1}(0)$ is a CR-function, then u has a continuous extension to $(b\mathcal{O})^+ \cup \mathcal{O}$ that is holomorphic on \mathcal{O} , whence u is a CR-function on all of $(b\mathcal{O})^+$.*

Proof. Let K be the compact set given by

$$K = \{z \in b\mathcal{O} : \operatorname{Re} z_1 = 0 \text{ or } u(z) = 0\},$$

and let $\mathcal{O}' = \mathcal{O} \setminus \widehat{K}$. We need to know that every CR-function v on $b\mathcal{O} \setminus K$ has a continuous extension, \tilde{v} into $\overline{\mathcal{O}} \setminus \widehat{K}$ that is holomorphic on \mathcal{O}' . This fact is essentially contained in the literature; we discuss the point in the Appendix at the end of the paper. We have that $|\tilde{v}|_{\overline{\mathcal{O}} \setminus \widehat{K}} \leq |v|_{b\mathcal{O} \setminus K}$.

Let u_1 be the restriction of u to $b\mathcal{O} \setminus K$, and denote by \tilde{u}_1 the holomorphic extension of u_1 to $\mathcal{O} \setminus \widehat{K}$. To begin with, we shall show that if $\{z^j\}_{j=1, \dots}$ is a sequence of points in \mathcal{O}' that tends to some point $p \in K \cap \{\operatorname{Re} z_1 > 0\}$, then $\tilde{u}_1(z^j)$ tends to 0. To see this, fix ψ , a function holomorphic on some neighborhood of \overline{D} that peaks at the point p so that $\psi(p) = 1$ and $|\psi| < 1$ on $\overline{D} \setminus \{p\}$. Let $\varepsilon > 0$, and fix k large enough that $|\psi^k u| \leq \varepsilon$ on $(b\mathcal{O})^+$. Such a k exists because $u(p) = 0$. We have $|\psi^k \tilde{u}_1| \leq \varepsilon$ on \mathcal{O}' , as the remark above implies. Since $\psi(p) = 1$, we get that $\limsup_{j \rightarrow \infty} |\tilde{u}_1(z^j)| \leq \varepsilon$, and we see that \tilde{u}_1 does tend to zero at the points of K in $\{\operatorname{Re} z_1 > 0\}$ as claimed.

We now show that if $\{z^j\}_{j \rightarrow \infty}$ is a sequence of points in \mathcal{O}' that converges to a point $p \in \widehat{K} \cap \mathcal{O}$, then $\tilde{u}_1(z^j)$ tends to zero. Let $\varepsilon > 0$, and fix $A > 0$ such that $e^{-A \operatorname{Re} p_1 / 4} \sup_{\mathcal{O}'} |\tilde{u}_1| < \varepsilon$. Let ω be a neighborhood of K such that for every $(z_1, z_2) \in \mathcal{O}' \cap \omega$, either $\operatorname{Re} z_1 < \operatorname{Re} p_1 / 2$ or $|e^{Az_1} \tilde{u}_1(z)| < \varepsilon$, which is possible because \tilde{u}_1 has limit zero at $K \cap \{\operatorname{Re} z_1 > 0\}$. The second lemma provides an open set $\Omega \subset \mathcal{O}'$ that contains the intersection of \mathcal{O}' with some neighborhood of \widehat{K} and on which we have

$$\sup_{(z_1, z_2) \in \Omega} |e^{Az_1} \tilde{u}_1| \leq \sup_{\mathcal{O}' \cap \omega} |e^{Az_1} \tilde{u}_1|.$$

For sufficiently large j , $z^j = (z_1^j, z_2^j) \in \Omega$, and $\text{Re } z_1^j > 3 \text{Re } p_1/4$, so we get

$$|\tilde{u}_1(z^j)| \leq \text{Max}(e^{A \text{Re } p_1/4} \sup |\tilde{u}_1|, \varepsilon) \leq \varepsilon,$$

and the assertion follows.

Finally, we define the function \tilde{u} by $\tilde{u} = u_1(z)$ on \mathcal{O}' and $\tilde{u}(z) = 0$ if $z \in \mathcal{O} \setminus \mathcal{O}'$. It follows from our remarks above that \tilde{u} is a continuous extension of u to $(b\mathcal{O})^+ \cup \mathcal{O}$ that is holomorphic on \mathcal{O} by virtue of the classical version of Radó's theorem. This completes the proof of Proposition 1' and so of Proposition 1.

III. PROOF OF PROPOSITION 2

The proof breaks into two parts. Denote by E the set of points in Σ where the Levi form of Σ vanishes. We will first prove that u , as given in the statement of Proposition 2, is a CR-function on $\Sigma \setminus E$. In terms of holomorphic extensions, this is related to the fact that one can get holomorphic extensions from holomorphic extensions along slices. Here, however, we shall work directly in terms of the CR-structure. Fix $p \in \Sigma \setminus E$, and let ρ be a defining function. We may choose coordinates so that $p = 0$ and $(\partial\rho/\partial\bar{z}_1)(0) \neq 0$. Denote by T_0^C the complex tangent space to Σ at 0. If we orient Σ properly, the Levi form of Σ , which we shall denote by \mathcal{L}_Σ , has at least one positive eigenvalue. Let $t = (t_2, \dots, t_N)$ be a vector in T_0^C that satisfies $\mathcal{L}_\Sigma(t) > 0$. Let M be the two-dimensional subspace spanned by the vectors t and $(1, 0, \dots, 0)$. We foliate a neighborhood of $0 \in \mathbb{C}^N$ by leaves parallel to M , and we apply Proposition 1 to the intersection of Σ with each of the leaves.* This shows that if we set

$$\overline{\mathcal{L}}_t = \sum_{j=2}^N \bar{t}_j \frac{\partial}{\partial \bar{z}_j} + \alpha(z) \frac{\partial}{\partial \bar{z}_1}$$

with $\alpha = -(\partial\rho/\partial\bar{z}_1) \sum_{j=2}^N \bar{t}_j \partial\rho/\partial\bar{z}_j$, which satisfies $\alpha(0) = 0$, then $\overline{\mathcal{L}}_t(u) = 0$ on a neighborhood of 0 in Σ . We can replace t by another vector $t' \in T_0^C$ close enough to t . Doing so shows that for a set $\{\overline{\mathcal{L}}_{t'}\}$ of generators for the tangential Cauchy-Riemann equations, $\overline{\mathcal{L}}_{t'}u = 0$ on a neighborhood of $0 \in \Sigma$.

To complete the proof of Proposition 2, we deal with the set E . Set $E' = \{z \in E : u(z) = 0\}$. By the hypothesis and what we have done above, u is continuous on Σ , the restriction of u to $\Sigma \setminus E'$ is a CR-function, and $u = 0$ on E' .

For every $\varepsilon > 0$ there exists a neighborhood E_ε of E' in Σ on which $|u| \leq \varepsilon$ and which is bounded (in Σ) by a piecewise smooth hypersurface in Σ of $(2N - 2)$ -dimensional area bounded independently of ε . Set $u_\varepsilon = u\chi_\varepsilon$ where

* There is a detail to be dealt with here, viz., to know that the restriction of a CR-function on Σ to the intersection of Σ with a transverse, two-dimensional subspace is again a CR-function. For continuously differentiable functions, the result is clear. The case of continuous CR-functions is correct as may be seen by invoking the approximation theorem of Baouendi and Treves [1]. An alternative, much simpler route to the same conclusion is given in the forthcoming paper [10].

χ_ε is the characteristic function of E_ε . If \bar{L} is a tangential Cauchy–Riemann operator defined on Σ , then when we apply \bar{L} to u_ε , we get only a jump term carried by bE_ε , the boundary of E_ε in Σ : $\bar{L}u_\varepsilon = uh_\varepsilon d\sigma_\varepsilon$ where $d\sigma_\varepsilon$ is the area measure on bE_ε , and h_ε is bounded in terms of the coefficients of \bar{L} . This is clear, for in terms of local coordinates $(\xi_1, \dots, \xi_{2n-1})$ on Σ , we have that $(\partial/\partial\xi_j)\chi_\varepsilon$ is the measure induced by the form

$$(-1)^{j+1} d\xi_1 \wedge \dots \wedge \widehat{d\xi_j} \wedge \dots \wedge d\xi_{2N-1}.$$

If we let ε approach zero, we get $\bar{L}u = 0$, as desired, since $|u| \leq \varepsilon$ on bE_ε .

This completes the proof.

If we use the fact that on Levi flat parts of Σ , Proposition 2 is an immediate consequence of the classical theorem of Radó, it would be enough to assume that the boundary of E in Σ is thin.

IV. AN APPLICATION

We end the paper with an application to the theory of removable singularities for bounded functions. Let D be a strongly pseudoconvex domain in \mathbb{C}^N , $N > 1$, with bD of class \mathcal{C}^2 . Let f be a function that is continuous on \bar{D} and holomorphic on D . We denote by $Z_b(f)$ the set of points in bD where f vanishes. This is a subset of bD with zero area, but it may have Hausdorff dimension $2N - 1$ as examples of Stensønes [2, 16] show. Let g be a bounded measurable function on bD that satisfies the tangential Cauchy–Riemann equations in the weak sense on $bD \setminus Z_b(f)$. We shall show that in fact g satisfies weakly the tangential Cauchy–Riemann equations on all of bD , and thus, that g is the boundary value function of a function bounded and holomorphic on D . Briefly put, $Z_b(f)$ is a removable set for bounded weak solutions of $\bar{\partial}_b$ on bD .

Consider first the case that the function g is continuous outside $Z_b(f)$. The function h defined to be zero on $Z_b(f)$ and defined to be fg on $bD \setminus Z_b(f)$ satisfies the tangential Cauchy–Riemann equations in $bD \setminus Z_b(f)$ and is continuous on bD . As it vanishes on the set $Z_b(f)$, our version of Radó’s theorem implies the existence of a function H holomorphic on D with the continuous boundary values h . The function $G = Hf^{-1}$ is meromorphic on D and assumes continuously the boundary values g almost everywhere on bD . We shall show that, in fact, the function G is holomorphic and bounded.

To see this, introduce the set $M_\delta = \{z \in D: |f(z)| = \delta\}$, and notice that it is fibered by the codimension one subvarieties $f^{-1}(\delta e^{i\theta}) \cap D$. In addition, let $D_\delta = D \setminus \{z: |f(z)| < \delta\}$. We have that $bD_\delta = M_\delta \cup \Gamma$ if $\Gamma = bD \setminus \{z: |f(z)| < \delta\}$. As M_δ is fibered by analytic varieties of positive dimension and as g is bounded uniformly on bD , say $|g| \leq 1$, we have $|G| \leq 1$ on D_δ by the maximum principle. This bound is correct for all choices of δ , so G is seen to be bounded on $D \setminus f^{-1}(0)$. Consequently, G extends to be holomorphic on

all of D , and it is also bounded. This disposes of the special case of the result in which g is assumed to be continuous.

Note that there is an alternative, functional-analytic way to see that the function G is bounded. It is bounded because every homomorphism from the algebra of bounded CR-functions defined on $bD \setminus Z_b(f)$, equipped with the supremum norm, into \mathbb{C} is of norm one. From this, $|G| \leq \sup_{bD \setminus Z_b(f)} |g|$.

The case that g is merely bounded reduces immediately to the case just considered as follows. The strict pseudoconvexity of D assures us that the function g extends locally into D from every point in $bD \setminus Z_b(f)$. Thus, if we denote by D_1 a domain obtained from D by pulling bD in slightly along $bD \setminus Z_b(f)$ and leaving it fixed along the set $Z_b(f)$, then D_1 is again strongly pseudoconvex, and the function g extends holomorphically into the crescent shaped domain $D \setminus D_1$ as a bounded function, say g_1 . We can arrange so that, in fact, g_1 is continuous on $bD_1 \setminus Z_b(f)$. Apply the result we have obtained for continuous functions to the restriction $g_1|(bD_1 \setminus Z_b(f))$ to find that g_1 continues holomorphically into D_1 as a bounded function. We may conclude that the function g continues through D as a bounded holomorphic function, as we wished to show.

APPENDIX

In the notation used in the proof of Proposition 1', we are to see that each CR-function v on $b\mathcal{O} \setminus K$ has a continuous extension, \tilde{v} , to $\overline{\mathcal{O}} \setminus \hat{K}$ that is holomorphic on \mathcal{O}' . There are two ways to see that this is so. First, let W denote a component of \mathcal{O}' . Then $bW = \Omega \cup E$, where Ω is an open set in bD and $E = bW \cap \hat{K}$. The set E has the convexity property that given $z \in \overline{W} \setminus E$, there is a polynomial p with

$$p(z) = 1 > \|p\|_E,$$

for $E \subset \hat{K}$, and $\overline{W} \setminus E \cap \hat{K} = \emptyset$. Moreover, $bW \setminus E = \Omega$ is connected, as follows from Alexander's theorem [0]: *If $X \subset b\mathbf{B}_N$ is compact, then $b\mathbf{B}_N \setminus X$ and $\mathbf{B}_N \setminus \hat{X}$ have the same number of components.* The proof Alexander gives for this applies equally well to an arbitrary convex domain in place of the ball. Granted the connectedness of Ω , our function v continues holomorphically into W by [7, Theorem 1]. As this is true for every component of \mathcal{O}' , we have the desired extension.

An alternative way to obtain what we need is to give a direct proof of the following way.

Proposition 3. *Let U be a bounded open set in \mathbb{C}^N , $N \geq 2$. Let $K \subset bU$ be a compact set such that U is strictly pseudoconvex at every point of the set $bU \setminus K$, which we take to be a smooth manifold of class C^2 . Assume that $\hat{K} \cap bU = K$. Then every continuous CR-function on $bU \setminus K$ has a continuous extension to $(bU \setminus K) \cup (U \setminus \hat{K})$ that is holomorphic on $U \setminus \hat{K}$.*

Here again, \hat{K} denotes the polynomially convex hull of the set K .

Proof. Let f be a continuous CR-function on $bU \setminus K$. By the strong pseudoconvexity of $bU \setminus K$, f extends locally into U along $bU \setminus K$. If χ is an infinitely differentiable function on $\mathbb{C}^N \setminus K$ with χ identically one on a neighborhood of $bU \setminus K$ and identically zero away from $bU \setminus K$, the function f^* given by $f^* = \chi f$ on the domain of f , 0 on $U \cap \{\chi = 0\}$ is a smooth function on U that agrees with f near $bU \setminus K$.

Define a $(0, 1)$ -form G on $\mathbb{C}^N \setminus \widehat{K}$ by letting $G = \bar{\partial} f^*$ on U , $G = 0$ on $\mathbb{C}^N \setminus (\widehat{K} \cup U)$.

The form G is smooth and is $\bar{\partial}$ -closed. We shall show that *there is a function u defined on $\mathbb{C}^N \setminus \widehat{K}$ with $\text{supp } u \subset \{z \in \mathbb{C}^N : |z| < R\}$ for some R such that $\bar{\partial} u = G$ on $\mathbb{C}^N \setminus \widehat{K}$.*

Notice that $\mathbb{C}^N \setminus (\widehat{K} \cup \bar{U})$ cannot have any bounded component. Indeed, if Y were such a component, bY would be strictly pseudoconcave at every point of $bY \setminus \widehat{K}$, and therefore for every polynomial P one would have: $\sup_Y |P| < \sup_{\widehat{K}} |P|$, clearly a contradiction with the definition of \widehat{K} . Then, Proposition 3 follows immediately. As u is holomorphic off $\widehat{K} \cup \bar{U}$ we have $u = 0$ on $bU \setminus K$, and thus $f^* - u$ gives the desired extension.

The existence of the function u is a consequence of the following fact:

Lemma. *Let X be a compact polynomial convex subset of \mathbb{C}^N , $N \geq 2$. Given a smooth $\bar{\partial}$ -closed $(0, 1)$ -form G on $W = \mathbb{C}^N \setminus X$, G with bounded support, there is a function u defined in W with bounded support that satisfies $\bar{\partial} u = G$ on W .*

Proof. Begin by fixing a strongly pseudoconvex neighborhood Ω of X with $b\Omega$ smooth and $\bar{\Omega}$ polynomially convex. We will first solve $\bar{\partial} u = G$ on $\mathbb{C}^N \setminus \Omega$.

Denote by v a smooth function on \mathbb{C}^N that satisfies $\bar{\partial}_b v = G$ on $b\Omega$. For $N \geq 3$, this is immediately possible [4]. For $N = 2$, more effort is required [4, 12]; we must verify that

$$\int_{b\Omega} hG \wedge dz_1 \wedge dz_2 = 0$$

for every smooth function h that is holomorphic on $\bar{\Omega}$. However, by virtue of the assumption of polynomial convexity and the Oka-Weil theorem, it suffices to verify this condition for polynomials h . If h is a polynomial, we can use Stokes's theorem to write for large $R > 0$ that

$$\int_{b\Omega} hG \wedge dz_1 \wedge dz_2 = \int_{|z|=R} hG \wedge dz_1 \wedge dz_2 = 0.$$

Thus, the desired v exists independently of the value of N .

Granted that there is a smooth function v on \mathbb{C}^N with $\bar{\partial}_b v = G$ on $b\Omega$, it follows [12, Remark, p. 125] that there is a smooth function on \mathbb{C}^N , again denoted by v , with $\bar{\partial} v = G$ on $b\Omega$.

It follows that we can define a continuous form G^* with compact support on \mathbb{C}^N by taking G^* to be G on $\mathbb{C}^N \setminus \Omega$ and to be $\bar{\partial} v$ on $\bar{\Omega}$. The form G^*

is continuous, and it is $\bar{\partial}$ -closed in the sense of currents: If θ is a smooth $(N, N-2)$ -form on \mathbb{C}^N with compact support, then

$$\int_{\mathbb{C}^N} \bar{\partial}\theta \wedge G = 0,$$

for this integral is the sum

$$\int_{\Omega} \bar{\partial}\theta \wedge G^* + \int_{\mathbb{C}^N \setminus \Omega} \bar{\partial}\theta \wedge G^*,$$

which vanishes by Stokes's theorem. It follows that there is a compactly supported continuous function u that is smooth off $b\Omega$ and that satisfies $\bar{\partial}u = G^*$ in the sense of distributions.

If we choose a sequence $\{\Omega_j\}_{j=1, \dots}$ of domains like Ω with $\Omega_1 \supset \supset \Omega_2 \supset \supset \dots$ and $\bigcap_j \Omega_j = X$, and if we let u_j satisfy $\bar{\partial}u_j = G$ on Ω_j , we can define u on $\mathbb{C}^N \setminus X$ by $u(z) = u_j(z)$ for large j : $u_j - u_{j+1}$ is holomorphic in the connected set $\mathbb{C}^N \setminus \Omega_j$ and has bounded support. Thus, $u_j = u_{j+1}$ throughout $\mathbb{C}^N \setminus \Omega_j$, and u is well defined. It satisfies $\bar{\partial}u = G$ and $\mathbb{C}^N \setminus K$.

The proof just given for Proposition 3 follows closely the lines of the proof of Theorem 2.3.2' of [5].

REFERENCES

0. H. Alexander, *A note on polynomial hulls*, Proc. Amer. Math. Soc. **33** (1972), 38–391.
1. M. S. Baouendi and F. Trèves, *A property of the functions and distributions annihilated by a locally integrable system of complex vector fields*, Ann. Math. (2) **113** (1981), 387–421.
2. J. E. Fornæss and B. Stensønes, *Lectures on counterexamples in several complex variables*, Princeton Univ. Press, Princeton, N.J., 1987, 247.
3. T. W. Gamelin, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, 1969, xiii+257.
4. G. M. Henkin, *The Lewy equation and analysis on pseudoconvex manifolds*, Russian Math. Surveys (3) **32** (1977), 59–130.
5. L. V. Hörmander, *An introduction to complex analysis in several complex variables*, Van Nostrand, Princeton, N.J., 1966.
6. P. H. Krief, *Sur les formes différentielles fermées à coefficients différentiables mais non continûment différentiables*, Ann. Polon. Math. **XLIV** (1984), 201–207.
7. G. Lupacoliu, *A theorem on holomorphic extension of CR-functions*, Pacific J. Math. **124** (1986), 177–191.
8. G. Lupacoliu, *Holomorphic continuation in several complex variables*, Pacific J. Math. **128** (1987), 117–125.
9. G. Lupacoliu, *Some global results on extensions of CR-objects in complex manifolds*, to appear.
10. G. Lupacoliu and E. L. Stout, *Removable singularities for $\bar{\partial}_b$* , to appear.
11. R. Narasimhan, *Several complex variables*, Univ. of Chicago Press, Chicago and London, 1971, x+174.
12. J.-P. Rosay, *Equation de Lewy-résolubilité globale de l'équation $\bar{\partial}_b = f$ sur la frontière de domaines faiblement pseudo-convexes de \mathbb{C}^2 (ou de \mathbb{C}^n)*, Duke Math. J. **49** (1982), 121–128.
13. H. Rossi, *The local maximum modulus principle*, Ann. Math. (2) **72** (1960), 1–11.

14. W. Rudin, *Function theory in the unit ball of C^n* , Springer-Verlag, New York, 1980, xiii+436.
15. Z. Słodkowski, *Analytic set-valued functions and spectra*, Math. Ann. **225** (1981), 363–386.
16. B. Stensønes (Henricksen), *A peak set of Hausdorff dimension $2n - 1$ for the algebra $A(\mathcal{D})$ in the boundary of a domain \mathcal{D} with \mathcal{C}^∞ -boundary in C^n* , Math. Ann. **289** (1982), 271–277.
17. G. Stolzenberg, *Polynomially and rationally convex sets*, Acta Math. **109** (1963), 259–289.
18. E. L. Stout, *A generalization of a theorem of Radó*, Math. Ann. **177** (1968), 339–340.
19. E. L. Stout, *The theory of uniform algebras*, Bogden and Quigley, Tarrytown-on-Hudson, 1971, x+509.
20. E. L. Stout, *Analytic continuation and boundary continuity of functions of several complex variables*, Proc. Roy. Soc. Edinburgh Sect. A **89** (1981), 63–74.
21. V. S. Vladimirov, *Les Fonctions de Plusieurs Variables Complexes*, Dunod, Paris, 1971, xv+338.

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