A DYNAMICAL SYSTEM ON $R^3$ WITH UNIFORMLY BOUNDED TRAJECTORIES AND NO COMPACT TRAJECTORIES

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Abstract. This paper contains an example of a rest point free dynamical system on $R^3$ with uniformly bounded trajectories, and with no circular trajectories. The construction is based on an example of a dynamical system described by P. A. Schweitzer, and on an example of a dynamical system on $R^3$ constructed previously by the authors.

INTRODUCTION

In order to answer S. Ulam’s question posed in the Scottish Book (see Problem 110 in [5 or 3]) we constructed in [2] a rest point free dynamical system $\Phi$ on $R^3$ with all trajectories uniformly bounded by a given $e > 0$. The function $f(x) = \Phi(1, x)$ provided a counterexample to Ulam’s problem. The dynamical system $\Phi$ uses elements similar to Fuller’s flow described in [1] (see also Wilson [6]), and hence $\Phi$ contains circular trajectories. The purpose of this paper is to eliminate all circular trajectories by modifying the example described in [2]. This is done with the help of Schweitzer's flow, constructed in [4] in order to solve the Seifert conjecture on closed integral curves of a nonvanishing vector field on $S^3$.

THE EXAMPLE

P. A. Schweitzer, see [4, pp. 393-394], based the construction of a “plug” with no circular trajectories on Denjoy’s vector field on a surface of a torus. A $C^1$ Denjoy vector field is included in the Appendix of [4]. We will now define a Schweitzer flow in such a way that the plug “stops” an open set of trajectories.

Assume the following notation: $R^n =$ Euclidean $n$-space, $S^1 =$ one-dimensional sphere, $T = S^1 \times S^1$, $d =$ distance function on $T$, $Z = C^1$ Denjoy’s vector field on $T$, $A =$exceptional minimal set of $Z$, and $\Phi(t, x) =$ trajectory of a dynamical system $\Phi$ passing through $x$.

Let $B$ be a compact proper subset of $T$, invariant under the dynamical system generated by $Z$, and such that $A \subset \text{Int } B$. Let $p_0 \in T - B$ be a point.
For \( i = 0, 1, 2 \), let \( D_i \subset T \) be a disc centered at \( p_0 \) with radius \( r_i \) such that \( r_0 < r_1 < r_2 \leq d(p_0, B) \). Let \( f: T \to R^1 \) be a \( C^\infty \) mapping such that \( f|D_1 \equiv 0 \), \( f|T - D_2 \equiv 1 \), and if \( q_1, q_2 \in D_2 - D_1 \) are such that \( d(q_1, p_0) < d(q_2, p_0) \), then \( f(q_1) < f(q_2) \). Let \( g: R^1 \to R^1 \) be an increasing \( C^\infty \) function such that \( g|(-\infty, 0] \equiv 0 \) and \( g|[1/6, \infty) \equiv 1 \). Define \( \alpha, \beta: R^1 \to R^1 \) by

\[
\alpha(s) = \begin{cases} 
0 & \text{if } s \leq \frac{1}{6}, \\
g(s - \frac{1}{6}) & \text{if } \frac{1}{6} \leq s \leq \frac{1}{3}, \\
g(\frac{1}{3} - s) & \text{if } \frac{1}{3} \leq s \leq \frac{1}{2}, \\
-\alpha(1 - s) & \text{if } s \geq \frac{1}{2}, 
\end{cases}
\]

\[
\beta(s) = \begin{cases} 
\alpha(s) & \text{if } s \leq \frac{1}{2}, \\
-\alpha(s) & \text{if } s \geq \frac{1}{2}.
\end{cases}
\]

Set \( M = T - D_0 \). Let \( V \) be a vector field defined on the manifold \( M \times R^1 \) by setting

\[
V((p, s)) = (f(p)Z(p)\alpha(s), 1 - f(p)\beta(s)).
\]

Notice that if \( V((p, s)) = (v_1, v_2) \), then \( V((p, 1 - s)) = (-v_1, v_2) \); and \( V((p, s)) = (0, 1) \) if either \( p \in D_1 - D_0, s \leq 1/6 \), or \( s \geq 5/6 \). The dynamical system \( \Phi \) generated by \( V \) has the following properties:

(i) for every point \( p \in B \) the trajectory containing the point \((p, 1)\) has its \( \alpha \)-limit points in the interior of the set \( M \times [0, 1] \), and the trajectory containing the point \((p, 0)\) has its \( \omega \)-limit points in the interior of the set \( M \times [0, 1] \),

(ii) for any number \( t > 0 \) and any point \((p, s) \in [M \times (R^1 - \{1/3, 2/3\})] \cup \text{Int}[(D_2 - D_0) \times R^1] \) if \( \Phi(t, (p, s)) = (q, r) \), then \( r > s \),

(iii) if \((p, 0)\) and \((q, 1)\) belong to the same trajectory, then \( p = q \),

(iv) \( \Phi \) contains no circular trajectories.

There exists a \( C^\infty \) embedding \( \mu: M \times [0, 1] \to R^3 \) such that for any \( p \in M \), the set \( \{p\} \times [0, 1] \) is contained in a line parallel to the \( z \)-axis, and for any \((p, s) \in M \times [0, 1] \), there exists a neighborhood \( N \) of \( p \) in \( M \) such that the projection onto the \( xy \)-plane restricted to the set \( \mu(N \times \{s\}) \) is an embedding, see [2]. We may also assume that \( \mu(M \times [0, 1]) \) is a subset of the cube \( \{(x, y, z) \in R^3: 0 \leq x, y, z \leq 1\} \).

There exists a \( C^1 \) vector field \( U \) defined on \( R^3 \) such that \( U \) and the dynamical system \( \psi \) generated by \( U \) satisfy the following conditions:

(i) \((x, y, z) \notin \mu(M \times [0, 1]) \), then \( U((x, y, z)) = (0, 0, 1) \),

(ii) for any trajectory \( \psi(t, (x, y, z)) \), the set \( \psi(t, (x, y, z)) \cap \mu(M \times [0, 1]) \) is contained in the set \( \mu(\Phi(t, (p, s))) \) for some \((p, s) \in M \times [0, 1] \),

(iii) for any trajectory \( \psi(t, (x, y, z)) \), the set \( \psi(t, (x, y, z)) \cap (R^3 - \mu(M \times [0, 1])) \) is contained in a line parallel to the \( z \)-axis.

Remark. No trajectory of \( \psi \) is compact, and if a trajectory of \( \psi \) passes through the points \((x_1, y_1, 0)\) and \((x_2, y_2, 1)\), then \((x_1, y_1) = (x_2, y_2)\).
There exists a nonempty subset of the plane \( Q = \{(x,y) \in \mathbb{R}^2: a \leq x \leq a + \delta, b \leq y \leq b + \delta\} \) such that if \((x,y) \in Q\), then the \(\alpha\)-limit points of \(\psi(t,(x,y,1))\) and the \(\omega\)-limit points of \(\psi(t,(x,y,0))\) are in the interior of the cube \(\{(x,y,z) \in \mathbb{R}^3: 0 \leq x,y,z \leq 1\}\).

Let \([x]\) denote the greatest integer less than or equal to \(x\). Let \(N_0\) be an integer greater than \(1/\delta\). We define a vector field \(W\) on \(R^3\) in the following way: if \(iN_0 + j \leq z < iN_0 + j + 1\), where \(i\) and \(j\) are integers, and \(0 \leq j < N_0\), then put
\[
W((x,y,z)) = U((x - i/N_0 - [x - i/N_0], y - j/N_0 - [y - j/N_0], z - [z])).
\]

Clearly, \(W\) is a \(C^1\) vector field defined on \(R^3\). Similarly as in [1], the trajectories of the dynamical system generated by \(W\) are uniformly bounded; every trajectory is contained in a box of dimensions \(3, 3, \) and \(N_0^2\), and hence it is bounded by \(\sqrt{18 + N_0^4}\).

**Remark 1.** By rescaling the example, the uniform bound for the diameter of the trajectories can be changed to an arbitrarily pre-assigned \(\varepsilon > 0\).

**Remark 2.** The vector field constructed in this paper is of class \(C^1\). The following question remains unanswered: Does there exist a \(C^\infty\) dynamical system on \(R^3\) with uniformly bounded trajectories and no compact trajectories?

**References**


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