ONE CRITERION FOR UNIVALENCY

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Abstract. In this note we give one criterion for the functions $f(z) = z + a_2 z^2 + \cdots$ to be univalent in $|z| < 1$.

Let $f(z)$ and $g(z)$ be analytic in the unit disc $U = \{z : |z| < 1\}$. We say that the function $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$, if $g(z)$ is univalent in $U$, $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

We can easily derive the following lemma from one earlier result by Miller and Mocanu ([1, Theorem 3, p. 190]). We can find more about differential subordinations in the same paper.

Lemma 1. If $p(z) = 1 + p_1 z + \cdots$ is analytic in $U$ and

$$zp'(z) < z,$$

then

$$p(z) \prec 1 + z$$

and the function $1 + z$ is the best dominant for the differential subordination (1).

We owe the next lemma to Ozaki and Nunokawa [2].

Lemma 2. Let the function $f(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be analytic in $U$ and

$$\Re \left\{ \frac{f(z)^2}{z^2 f'(z)} \right\} \geq \frac{1}{2} \quad (z \in U),$$

(which is equivalent to $|z^2 f'(z)/f(z)^2 - 1| \leq 1$ $(z \in U)$; then $f(z)$ is univalent in $U$.

By using these lemmas we obtain
Theorem. Let the function $f(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be analytic in $U$ with $f(z)/z \neq 0$, $0 < |z| < 1$, and

(2) \[ \left| \left( \frac{z}{f(z)} \right)' \right| \leq 1 \quad (z \in U); \]

then $f(z)$ is univalent in $U$.

Proof. If we put

$$p(z) = \frac{z}{f(z)} - z \left( \frac{z}{f(z)} \right)' ,$$

then we get

$$p'(z) = -z \left( \frac{z}{f(z)} \right)''$$

and

$$p(z) = \frac{z^2 f'(z)}{f(z)^2}.$$ 

For such $p(z)$ from Lemma 1 we have that the following implication

(3) \[- z^2 \left( \frac{z}{f(z)} \right)'' < z \Rightarrow \frac{z^2 f'(z)}{f(z)^2} < 1 + z\]

is true and the function $1 + z$ is the best dominant.

On the other hand from (2) we obtain

$$\left| z^2 \left( \frac{z}{f(z)} \right)'' \right| \leq |z|^2 < 1 \quad (z \in U),$$

which implies

$$z^2 \left( \frac{z}{f(z)} \right)'' < z$$

and so

$$- z^2 \left( \frac{z}{f(z)} \right)'' < z .$$

Therefore, from (3) we have

$$\frac{z^2 f'(z)}{f(z)^2} < 1 + z ,$$

or

$$\left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| < 1 .$$

Finally, from Lemma 2 we conclude that $f(z)$ is univalent in $U$. 

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