EXACHE HYPERININVARIANT SUBSPACE FOR MULTIPLICATION OPERATOR IS SPECTRAL

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Abstract. We consider multiplication operators on general separable complex $L^p$-spaces, for $1 < p < +\infty$, and obtain the result announced in the title. Moreover, a result of Douglas and Pearcy on normal operators is given an alternate proof.

In this paper, our notation is standard. We refer the reader to [1] and [4]. We consider only complex spaces $L^p(X,\mathcal{A},\mu,\mathbb{C})$ for $1 < p < +\infty$. We also simply denote $L^p(X,\mathcal{A},\mu,\mathbb{C})$ by $L^p(X,\mathcal{A},\mu)$ or $L^p(X,\mu)$ in the circumstances where there is no confusion.

Let $\phi \in L^\infty(X,\mu)$. The multiplication operator corresponding to $\phi$ is the bounded operator $M_\phi$ on $L^p(X,\mu)$ defined by $(M_\phi f)(x) = \phi(x)f(x)$ for all $f \in L^p(X,\mu)$. For a Borel set $S$ of $\mathbb{C}$, define $E^c(S) = M_1\circ\phi$, where by $1_S$ we always denote the characteristic function corresponding to a set $S$. Following Dunford [3], $E^c(\cdot)$ is a spectral measure which makes $M_\phi$ a spectral operators. For a spectral operator, there is the following remarkable theorem of Fuglede and Dunford (its proof can be found in [5]).

F-D Theorem. If $A$ is a spectral operator with spectral measure $E(\cdot)$ and if $AB = BA$, then $BE(S) = E(S)B$ for all Borel sets $S$.

As a result of this theorem, we have

Corollary. Let $E(\cdot)$ be any spectral measure for $M_\phi$. If $\mathcal{M}$ is the range of $E(S)$ for some Borel set $S$, then $\mathcal{M}$ is a hyperinvariant subspace for $M_\phi$. Thus each nonscalar multiplication operator has a nontrivial hyperinvariant subspace.

Proof. Trivial.

From this corollary we see that the range of $E^c(S)$ for each Borel set $S$ is a hyperinvariant subspace for $M_\phi$. We will next see that each hyperinvariant subspace for $M_\phi$ can be written in this form whenever $M_\phi$ is defined on a separable space. Because of this, by $E^c(\cdot)$ we always mean this special spectral...
measure defined as at the beginning. Moreover, by a subspace we always mean a linear subspace and we call a closed subspace $\mathcal{M}$ of $L^p(X, \mu)$ a spectral subspace if $\mathcal{M} = E^c(S)L^p(X, \mu)$ for some Borel set of $\mathcal{C}$. We also denote the algebra of all multiplication operators on $L^p(X, \mu)$ by $L^\infty$, i.e.

$$L^\infty = \{M_\phi: \phi \in L^\infty(X, \mu)\}.$$

Let us start with a converse of the F-D Theorem for multiplication operators.

**Lemma 1.** If $B$ commutes with all $E^c(S)$, then $B$ commutes with $M_\phi$.

**Proof.** This is trivial by the definition of the spectral integral.

Now we pass to the structure of hyperinvariant subspaces for multiplication operators.

**Lemma 2.** Let $\mathcal{M}$ be a closed separable subspace of $L^p(X, \mathcal{A}, \mu)$ for $1 \leq p < +\infty$. If $\mathcal{M}$ is invariant for $L^\infty$, i.e. $M_\phi \mathcal{M} \subseteq \mathcal{M}$ for all $M_\phi \in L^\infty$, then there exists some $A \in \mathcal{A}$ such that $\mathcal{M} = 1_A \cdot L^p(X, \mathcal{A}, \mu)$.

**Proof.** Since $\mathcal{M}$ is separable, a theorem of Ando (see [4], page 152, Lemma 1) implies that there exists an $f \in \mathcal{M}$ with maximal support $A$ of all functions in $\mathcal{M}$. It follows that $A \in \mathcal{A}$ and $\mathcal{M} \subseteq 1_A \cdot L^p(X, \mu)$. We must show that $\mathcal{M} = 1_A \cdot L^p(X, \mu)$. To this end, assume $A \supseteq B \in \mathcal{A}$ with $\mu(B) < +\infty$. Let

$$A_n = \{x \in X: \frac{1}{n} \leq |f(x)| \leq n\} \quad \text{for all } n \geq 1.$$

Put $\phi_n = f^{-1} \cdot 1_A \cdot 1_B$. Then $\phi_n \in L^\infty(X, \mu)$; and hence $1_{B \cap A_n} = \phi_n \cdot f \in \mathcal{M}$. We have that $\mu(B \setminus B \cap A_n) \to 0$ as $n \to \infty$. Since $\mathcal{M}$ is closed, this implies that $1_B \in \mathcal{M}$. And thus $1_B \cdot L^\infty(X, \mu) \subseteq \mathcal{M}$ for all $A \supseteq B \in \mathcal{A}$ with $\mu(B) < +\infty$.

It is now routine to check that $\mathcal{M} = 1_A \cdot L^p(X, \mu)$ since $\mathcal{M}$ is closed and since the span of subspaces $1_B \cdot L^p(X, \mu)$ for $A \supseteq B \in \mathcal{A}$ with $\mu(B) < +\infty$ is dense in $1_A \cdot L^p(X, \mu)$. This completes the proof.

To continue, we need one more notion. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $\phi \in L^\infty(X, \mathcal{A}, \mu)$. Let $\mathcal{A}_0 = \{\phi^{-1}(S): S$ a Borel set of $\mathcal{C}\}$. It is easily seen that $\mathcal{A}_0$ is also a sub-sigma-algebra of $\mathcal{A}$. We regard $(X, \mathcal{A}_0, \mu)$ as a finite measure space. Let $A \in \mathcal{A}$ be given. By the well-known Radon-Nikodym theorem we may define the conditional expectation operator $E_A$, for the measure $\mu$ relative to $\mathcal{A}_0$. $E_A$ is uniquely determined by the equation

$$\int_B h \cdot 1_A d\mu = \int_B (E_A h) d\mu \quad (B \in \mathcal{A}_0)$$

for $h \in L^1(X, \mathcal{A}, \mu)$ and by the condition that $E_A h$ is $\mathcal{A}_0$-measurable. This class of operators $\{E_A : A \in \mathcal{A}\}$ has the following interesting properties.

**Lemma 3.** (1) Each $E_A$ is a positive linear operator on $L^p(X, \mu)$, $1 \leq p \leq \infty$ with $\|E_A\| \leq 1$. In particular, $0 \leq E_A(h) \leq \|h\|_\infty$ for all $h \in L^\infty(X, \mu)$ with $0 \leq h$.

(2) Each $E_A$ commutes with $M_\phi$, i.e. $E_A M_\phi = M_\phi E_A$. 
Proof. (1) We may write $E_A = E_X \circ M_1$, where $E_X$ is the usual conditional expectation operator determined by the sub-sigma-algebra $\mathcal{A}_0$ (cf. [1]). Since both $E_X$ and $M_1$ are positive contractions on each $L^p(X, \mu)$, $1 \leq p \leq \infty$, as is well known, (1) follows.

(2) By Lemma 1, it suffices to show that $E_A$ commutes with all $E^c(S)$ for Borel sets $S$ of $\mathbb{C}$. Fix such an $S$. Then $E^c(S) = M_1^{-1}(S)$ and $\phi^{-1}(S) \in \mathcal{A}_0$. By the definition of $E_A$, for all $B \in \mathcal{A}_0$,

$$\int_B E^c(S)(E_A h) d\mu = \int_{B \cap \phi^{-1}(S)} (E_A h) d\mu = \int_{B \cap \phi^{-1}(S)} h \cdot 1_A d\mu$$

$$= \int_B (E^c(S)h) \cdot 1_A d\mu = \int_B E_A(E^c(S)h) d\mu,$$

for all $h \in L^1(X, \mathcal{A}, \mu)$. Since both $E^c(S)(E_A h)$ and $E_A(E^c(S)h)$ are $\mathcal{A}_0$-measurable, we conclude from the above that $E_A E^c(S) = E^c(S)E_A$. This finishes the proof.

Lemma 4. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $1 \leq p \leq +\infty$. Let $\phi \in L^\infty(X, \mu)$ and $M = 1_A \cdot L^p(X, \mu)$ for some $A \in \mathcal{A}$. Then $M$ is hyperinvariant for $M_\phi$ iff $A = \phi^{-1}(S)$ for some Borel set $S$, i.e. $M$ is a spectral subspace.

Proof. Assume that $M = 1_A \cdot L^p(X, \mu)$ is hyperinvariant for $M_\phi$. By Lemma 3 again, $E_A$ is a bounded operator which commutes with $M_\phi$. So $E_A(M) \subseteq M$. In particular, there exists $f_0 \in L^p(X, \mu)$ such that $E_A(1_A) = 1_A \cdot f_0$. Let $f_1 = E_A(1_A)$. By Lemma 3, again $0 \leq f_1 \leq 1$ a.e. Note that

$$\int_X (1-f_1) \cdot 1_A d\mu = \int_X 1_A d\mu - \int_X (E_A 1_A) d\mu = 0,$$

and so $(1-f_1) \cdot 1_A = 0$ a.e. Hence $1_A = 1_A \cdot f_1 = E_A(1_A)$ is $\mathcal{A}_0$-measurable. This implies that $A = \phi^{-1}(S)$ for some Borel set $S$. The converse is the corollary to the F-D Theorem.

Lemma 5. Let $(X, \mu)$ be a finite measure space and $1 \leq p < +\infty$. Let $\phi \in L^\infty(X, \mu)$ and let $M_\phi$ be the multiplication operator on $L^p(X, \mu)$. Then a closed separable subspace $M$ of $L^p(X, \mu)$ is hyperinvariant for $M_\phi$ iff $M$ is spectral, i.e. $M = E^c(S) L^p(X, \mu)$ for some Borel set $S$.

Proof. Use Lemma 2 and Lemma 4 while observing that each element of $L^\infty$ commutes with $M_\phi$.

For a Banach space $Z$ and a bounded linear operator $T$ on $Z$, let $\text{com}(T)$ be the commutant of $T$, i.e. the set of all bounded linear operators on $Z$ which commute with $T$. If $V: Z \rightarrow Y$ is an onto-isomorphism, then $\text{com}(VTV^{-1}) = V \cdot \text{com}(T) \cdot V^{-1}$.

Next, we state our main theorem.
Theorem 1. Let \( 1 \leq p < +\infty \) and \((X, \mu)\) be a measure space such that the space \( L^p(X, \mu) \) is separable. Let \( \phi \in L^\infty(X, \mu) \) and \( M_\phi \) be the multiplication
operator on \( L^p(X, \mu) \). Then a closed subspace \( \mathcal{M} \) of \( L^p(X, \mu) \) is hyperinvariant
for \( M_\phi \) iff \( \mathcal{M} \) is spectral, i.e. \( \mathcal{M} = E^c(S)L^p(X, \mu) \) for some Borel set \( S \).

Proof. Since \( L^p(X, \mu) \) is separable, by the same theorem of Ando used in the
proof of Lemma 2 above, one can easily build a finite measure \( \nu \) on \( X \) such
that \( \nu \) is equivalent to \( \mu \). Define \( V: L^p(X, \mu) \to L^p(X, \nu) \) by
\[
Vf = f \left( \frac{d\mu}{d\nu} \right)^{1/p} \quad \text{for all } f \in L^p(X, \mu).
\]
Then \( V \) is an onto-isometric isomorphism such that \( VM_\phi V^{-1} \) is the multipli-
cation operator \( M_\phi \) on \( L^p(X, \nu) \). From the previous remark, we immediately
obtain that \( \mathcal{M} \) is hyperinvariant for \( M_\phi \) on \( L^p(X, \mu) \) iff \( V\mathcal{M} \) is also hyper-
invariant for \( M_\phi \) on \( L^p(X, \nu) \). Use Lemma 5 while observing the definition
of \( V \), the latter assertion is equivalent to
\[
\mathcal{M} = E^c(S)L^p(X, \mu) \quad \text{for some Borel set } S.
\]
This completes the proof.

We give two applications of Theorem 1.

Corollary 1. Suppose \( 1 < p \leq 2 < +\infty \). Let \( L^p(X, \mu) \) be separable and let
\( \phi \in L^\infty(X, \mu) \). Let \( E^c(\cdot) \) be the special spectral measure corresponding to \( M_\phi \) on
the space \( L^p(X, \mu) \). Then \( E^c(\cdot) \) is maximal in the sense that if \( E(\cdot) \) is
another spectral measure for \( M_\phi \) on \( L^p(X, \mu) \) iff \( V\mathcal{M} \) is also hyper-
invariant for \( M_\phi \) on \( L^p(X, \nu) \). Use Lemma 5 while observing the definition
of \( V \), the latter assertion is equivalent to
\[
\mathcal{M} = E^c(S)L^p(X, \mu) \quad \text{for some Borel set } S.
\]
This completes the proof.

Corollary 2 (Douglas-Pearcy [2]). If \( A \) is a normal operator on the separable
Hilbert space \( H \) with spectral measure \( \{E_\lambda\} \), then \( \mathcal{M} \) is hyperinvariant for \( A \)
iff \( \mathcal{M} = E(S)H \) for some Borel set \( S \) of \( C \).

Proof. By the spectral theorem, we may assume that \( A = M_\phi \) on \( L^2(X, \mu) \) for
some measurable space \( (X, \mu) \). Also, since \( H \) is separable by hypothesis, so is
\( L^2(X, \mu) \). The result now follows from Theorem 1.
Note. This result was proved by Douglas and Pearcy [2] using facts from the theory of von Neumann algebras. Our proof seems to be more elementary and comes almost directly from the spectral theorem.

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