PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 106, Number 4, August 1989

EACH HYPERINVARIANT SUBSPACE FOR MULTIPLICATION OPERATOR IS SPECTRAL

HUANG SENZHONG

(Communicated by Paul S. Muhly)

ABSTRACT. We consider multiplication operators on general separable complex L^p-spaces, for $1 \leq p < +\infty$, and obtain the result announced in the title. Moreover, a result of Douglas and Pearcy on normal operators is given an alternate proof.

In this paper, our notation is standard. We refer the reader to [1] and [4]. We consider only complex spaces $L^p(X, \mathscr{A}, \mu, \mathbb{C})$ for $1 \leq p \leq +\infty$. We also simply denote $L^p(X, \mathscr{A}, \mu, \mathbb{C})$ by $L^p(X, \mathscr{A}, \mu)$ or $L^p(X, \mu)$ in the circumstances where there is no confusion.

Let $\phi \in L^\infty(X, \mu)$. The multiplication operator corresponding to ϕ is the bounded operator M_ϕ on $L^p(X, \mu)$ defined by $(M_\phi f)(x) = \phi(x)f(x)$ for all $f \in L^p(X, \mu)$. For a Borel set S of $\mathbb{C}$, define $E^c(S) = M_{1_S \circ \phi}$, where by 1_S we always denote the characteristic function corresponding to a set S. Following Dunford [3], $E^c(\cdot)$ is a spectral measure which makes M_ϕ a spectral operators. For a spectral operator, there is the following remarkable theorem of Fuglede and Dunford (its proof can be found in [5]).

F-D Theorem. *If A is a spectral operator with spectral measure $E(\cdot)$ and if $AB = BA$, then $BE(S) = E(S)B$ for all Borel sets S.*

As a result of this theorem, we have

Corollary. *Let $E(\cdot)$ be any spectral measure for M_ϕ. If $\mathscr{M}$ is the range of $E(S)$ for some Borel set S, then $\mathscr{M}$ is a hyperinvariant subspace for M_ϕ. Thus each nonscalar multiplication operator has a nontrivial hyperinvariant subspace.*

Proof. Trivial.

From this corollary we see that the range of $E^c(S)$ for each Borel set S is a hyperinvariant subspace for M_ϕ. We will next see that each hyperinvariant subspace for M_ϕ can be written in this form whenever M_ϕ is defined on a separable space. Because of this, by $E^c(\cdot)$ we *always* mean this special spectral

Received by the editors November 24, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47A15, 47B37.
Key words and phrases. Hyperinvariant subspace.

measure defined as at the beginning. Moreover, by a subspace we always mean a linear subspace and we call a closed subspace $\mathscr{M}$ of $L^p(X, \mu)$ a spectral subspace if $\mathscr{M} = E^c(S)L^p(X, \mu)$ for some Borel set of $\mathbb{C}$. We also denote the algebra of all multiplication operators on $L^p(X, \mu)$ by $\mathscr{L}^\infty$, i.e.

$$\mathscr{L}^\infty = \{M_\phi : \phi \in L^\infty(X, \mu)\}.$$

Let us start with a converse of the F-D Theorem for multiplication operators.

Lemma 1. *If B commutes with all $E^c(S)$, then B commutes with M_ϕ.*

Proof. This is trivial by the definition of the spectral integral.

Now we pass to the structure of hyperinvariant subspaces for multiplication operators.

Lemma 2. *Let $\mathscr{M}$ be a closed separable subspace of $L^p(X, \mathscr{A}, \mu)$ for $1 \le p < +\infty$. If $\mathscr{M}$ is invariant for $\mathscr{L}^\infty$, i.e. $M_\phi \mathscr{M} \subseteq \mathscr{M}$ for all $M_\phi \in \mathscr{L}^\infty$, then there exists some $A \in \mathscr{A}$ such that $\mathscr{M} = 1_A \cdot L^p(X, \mathscr{A}, \mu)$.*

Proof. Since $\mathscr{M}$ is separable, a theorem of Ando (see [4], page 152, Lemma 1) implies that there exists an $f \in \mathscr{M}$ with maximal support A of all functions in $\mathscr{M}$. It follows that $A \in \mathscr{A}$ and $\mathscr{M} \subseteq 1_A \cdot L^p(X, \mu)$. We must show that $\mathscr{M} = 1_A \cdot L^p(X, \mu)$. To this end, assume $A \supseteq B \in \mathscr{A}$ with $\mu(B) < +\infty$. Let

$$A_n = \{x \in X : \frac{1}{n} \le |f(x)| \le n\} \quad \text{for all } n \ge 1.$$

Put $\phi_n = f^{-1} \cdot 1_A \cdot 1_B$. Then $\phi_n \in L^\infty(X, \mu)$; and hence $1_{B \cap A_n} = \phi_n \cdot f \in \mathscr{M}$. We have that $\mu(B \backslash B \cap A_n) \to 0$ as $n \to \infty$. Since $\mathscr{M}$ is closed, this implies that $1_B \in \mathscr{M}$. And thus $1_B \cdot L^\infty(X, \mu) \subseteq \mathscr{M}$ for all $A \supseteq B \in \mathscr{A}$ with $\mu(B) < +\infty$. It is now routine to check that $\mathscr{M} = 1_A \cdot L^p(X, \mu)$ since $\mathscr{M}$ is closed and since the span of subspaces $1_B \cdot L^\infty(X, \mu)$ for $A \supseteq B \in \mathscr{A}$ with $\mu(B) < +\infty$ is dense in $1_A \cdot L^p(X, \mu)$. This completes the proof.

To continue, we need one more notion. Let $(X, \mathscr{A}, \mu)$ be a finite measure space and let $\phi \in L^\infty(X, \mathscr{A}, \mu)$. Let $\mathscr{A}_0 = \{\phi^{-1}(S) : S \text{ a Borel set of } \mathbb{C}\}$. It is easily seen that $\mathscr{A}_0$ is also a sub-sigma-algebra of $\mathscr{A}$. We regard $(X, \mathscr{A}_0, \mu)$ as a finite measure space. Let $A \in \mathscr{A}$ be given. By the well-known Radon-Nikodym theorem we may define the *conditional expectation operator* E_A, for the measure μ relative to $\mathscr{A}_0$. E_A is uniquely determined by the equation

$$\int_B h \cdot 1_A \, d\mu = \int_B (E_A h) \, d\mu \qquad (B \in \mathscr{A}_0)$$

for $h \in L^1(X, \mathscr{A}, \mu)$ and by the condition that $E_A h$ is $\mathscr{A}_0$-measurable. This class of operators $\{E_A : A \in \mathscr{A}\}$ has the following interesting properties.

Lemma 3. (1) *Each E_A is a positive linear operator on $L^p(X, \mu)$, $1 \le p \le \infty$ with $\|E_A\| \le 1$. In particular, $0 \le E_A(h) \le \|h\|_\infty$ for all $h \in L^\infty(X, \mu)$ with $0 \le h$.*

(2) *Each E_A commutes with M_ϕ, i.e. $E_A M_\phi = M_\phi E_A$.*

Proof. (1) We may write $E_A = E_X \circ M_{1_A}$, where E_X is the usual conditional expectation operator determined by the sub-sigma-algebra $\mathscr{A}_0$ (cf. [1]). Since both E_X and M_{1_A} are positive contractions on each $L^p(X, \mu)$, $1 \le p \le \infty$, as is well known, (1) follows.

(2) By Lemma 1, it suffices to show that E_A commutes with all $E^c(S)$ for Borel sets S of $\mathbb{C}$. Fix such an S. Then $E^c(S) = M_{1_{\phi^{-1}(S)}}$ and $\phi^{-1}(S) \in \mathscr{A}_0$. By the definition of E_A, for all $B \in \mathscr{A}_0$,

$$\begin{aligned}\int_B E^c(S)(E_A h)\, d\mu &= \int_{B\cap\phi^{-1}(S)} (E_A h)\, d\mu = \int_{B\cap\phi^{-1}(S)} h \cdot 1_A\, d\mu \\ &= \int_B (E^c(S)h) \cdot 1_A\, d\mu = \int_B E_A(E^c(S)h)\, d\mu,\end{aligned}$$

for all $h \in L^1(X, \mathscr{A}, \mu)$. Since both $E^c(S)(E_A h)$ and $E_A(E^c(S)h)$ are $\mathscr{A}_0$-measurable, we conclude from the above that $E_A E^c(S) = E^c(S) E_A$. This finishes the proof.

Lemma 4. *Let $(X, \mathscr{A}, \mu)$ be a finite measure space and $1 \le p \le +\infty$. Let $\phi \in L^\infty(X, \mu)$ and $\mathscr{M} = 1_A \cdot L^p(X, \mu)$ for some $A \in \mathscr{A}$. Then $\mathscr{M}$ is hyperinvariant for M_ϕ iff $A = \phi^{-1}(S)$ for some Borel set S, i.e. $\mathscr{M}$ is a spectral subspace.*

Proof. Assume that $\mathscr{M} = 1_A \cdot L^p(X, \mu)$ is hyperinvariant for M_ϕ. By Lemma 3 again, E_A is a bounded operator which commutes with M_ϕ. So $E_A(\mathscr{M}) \subseteq \mathscr{M}$. In particular, there exists $f_0 \in L^p(X, \mu)$ such that $E_A(1_A) = 1_A \cdot f_0$. Let $f_1 = E_A(1_A)$. By Lemma 3, again $0 \le f_1 \le 1$ a.e. Note that

$$\int_X (1 - f_1) \cdot 1_A\, d\mu = \int_X 1_A\, d\mu - \int_X (E_A 1_A)\, d\mu = 0,$$

and so $(1 - f_1) \cdot 1_A = 0$ a.e. Hence $1_A = 1_A \cdot f_1 = E_A(1_A)$ is $\mathscr{A}_0$-measurable. This implies that $A = \phi^{-1}(S)$ for some Borel set S. The converse is the corollary to the F-D Theorem.

Lemma 5. *Let (X, μ) be a finite measure space and $1 \le p < +\infty$. Let $\phi \in L^\infty(X, \mu)$ and let M_ϕ be the multiplication operator on $L^p(X, \mu)$. Then a closed separable subspace $\mathscr{M}$ of $L^p(X, \mu)$ is hyperinvariant for M_ϕ iff $\mathscr{M}$ is spectral, i.e. $\mathscr{M} = E^c(S) L^p(X, \mu)$ for some Borel set S.*

Proof. Use Lemma 2 and Lemma 4 while observing that each element of $\mathscr{L}^\infty$ commutes with M_ϕ.

For a Banach space Z and a bounded linear operator T on Z, let $\operatorname{comt}(T)$ be the commutant of T, i.e. the set of all bounded linear operators on Z which commute with T. If $V: Z \to Y$ is an onto-isomorphism, then $\operatorname{comt}(VTV^{-1}) = V \cdot \operatorname{comt}(T) \cdot V^{-1}$.

Next, we state our main theorem.

Theorem 1. *Let $1 \leq p < +\infty$ and (X, μ) be a measure space such that the space $L^p(X, \mu)$ is separable. Let $\phi \in L^\infty(X, \mu)$ and M_ϕ be the multiplication operator on $L^p(X, \mu)$. Then a closed subspace $\mathscr{M}$ of $L^p(X, \mu)$ is hyperinvariant for M_ϕ iff $\mathscr{M}$ is spectral, i.e. $\mathscr{M} = E^c(S)L^p(X, \mu)$ for some Borel set S.*

Proof. Since $L^p(X, \mu)$ is separable, by the same theorem of Ando used in the proof of Lemma 2 above, one can easily build a finite measure ν on X such that ν is equivalent to μ. Define $V: L^p(X, \mu) \to L^p(X, \nu)$ by

$$Vf = f \cdot \left(\frac{d\mu}{d\nu}\right)^{1/p} \quad \text{for all } f \in L^p(X, \mu).$$

Then V is an onto-isometric isomorphism such that $VM_\phi V^{-1}$ is the multiplication operator M_ϕ on $L^p(X, \nu)$. From the previous remark, we immediately obtain that $\mathscr{M}$ is hyperinvariant for M_ϕ on $L^p(X, \mu)$ iff $V\mathscr{M}$ is also hyperinvariant for M_ϕ on $L^p(X, \nu)$. Use Lemma 5 while observing the definition of V, the latter assertion is equivalent to

$$\mathscr{M} = E^c(S)L^p(X, \mu) \quad \text{for some Borel set } S.$$

This completes the proof.

We give two applications of Theorem 1.

Corollary 1. *Suppose $1 \leq p \neq 2 < +\infty$. Let $L^p(X, \mu)$ be separable and let $\phi \in L^\infty(X, \mu)$. Let $E^c(\cdot)$ be the special spectral measure corresponding to M_ϕ on the space $L^p(X, \mu)$. Then $E^c(\cdot)$ is maximal in the sense that if $E(\cdot)$ is another spectral measure for M_ϕ in the sense of Dunford* [3] *with contractive projections, then the range of $E(\cdot)$ is contained in that of $E^c(\cdot)$.*

Proof. Fix a Borel set S. Let $\mathscr{M} = E(S)L^p(X, \mu)$. By the corollary to the F-D Theorem, $\mathscr{M}$ is hyperinvariant for M_ϕ. Now Theorem 1 implies that $\mathscr{M} = E^c(S_1)L^p(X, \mu)$ for some Borel set S_1. It follows that $E(S)E^c(\mathbb{C}\backslash S_1) = 0$ since $E(S)$ commutes with $E^c(\mathbb{C}\backslash S_1)$ by the F-D Theorem. We consider the space $L^p(X, \mu)$ as a complex Banach lattice, and we refer the reader to [4] for a general theory of Banach lattices. We have, then, that $\mathscr{M}^{\mathrm{II}} = \mathscr{M}$, and $E^c(S_1)$ is the unique band projection on $\mathscr{M}^{\mathrm{II}}$. Since $E(S)$ is a contractive projection, a classical theorem (see [4], page 160, Theorem 2) implies that $E(S)E^c(S_1) = E^c(S_1)$. Combining these we finally obtain $E(S) = E^c(S_1)$. This finishes the proof.

Corollary 2 (Douglas-Pearcy [2]). *If A is a normal operator on the separable Hilbert space H with spectral measure $\{E_\lambda\}$, then $\mathscr{M}$ is hyperinvariant for A iff $\mathscr{M} = E(S)H$ for some Borel set S of $\mathbb{C}$.*

Proof. By the spectral theorem, we may assume that $A = M_\phi$ on $L^2(X, \mu)$ for some measurable space (X, μ). Also, since H is separable by hypothesis, so is $L^2(X, \mu)$. The result now follows from Theorem 1.

Note. This result was proved by Douglas and Pearcy [2] using facts from the theory of von Neumann algebras. Our proof seems to be more elementary and comes almost directly from the spectral theorem.

Acknowledgment

The author wishes to thank the referee for his valuable advice and meticulous help.

References

1. J. Diestel and J. J. Uhl, Jr., *Vector measures*, Amer. Math. Soc., Providence, R. I., 1977.
2. R. G. Douglas and C. Pearcy, *On a topology for invariant subspaces*, J. Funct. Anal. **2** (1968), 323–341.
3. N. Dunford, *Spectral operators*, Pacific J. Math. **4** (1954), 321–354.
4. H. E. Lacey, *The isometric theory of classical Banach spaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1974.
5. H. Radjavi and P. Rosenthal, *Hyperinvariant subspaces for spectral and n-normal operators*, Acta Sci. Math. (Szeged) **32** (1971), 121–126.
6. ——, *Invariant subspaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1973.

Department of Mathematics, Nankai University, Tianjin City, People's Republic of China