PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 106, Number 4, August 1989

NOWHERE DENSE P-SUBSETS OF ω^*

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(Communicated by Dennis Burke)

ABSTRACT. It is relatively consistent with ZFC that no nowhere dense P-subset of $\omega*$ is homeomorphic to the space ω^* itself.

By $P(\omega)$ we denote the Boolean algebra of all subsets of ω. Whenever we use the word 'ideal' we mean a proper ideal $I \subset P(\omega)$ such that I contains Fin—the ideal of finite subsets of ω. An ideal I is called a *p-ideal*, if for every countable subset $\{A_n: n \in \omega\} \subset I$ there exists a $B \in I$ such that $A_n - B \in Fin$ for every n. An ideal I is called *tall*, if for every infinite $B \subset \omega$ there exists an infinite $A \in I$ such that $A \subset B$. For instance, Fin is a p-ideal, but not tall.

The Stone space of $P(\omega)/Fin$ is ω^*, i.e. $\beta\omega - \omega$. A subset Y of a topological space X is called a *P-set*, if for every countable family U of open supersets of Y there exists an open V so that $Y \subseteq V \subseteq \bigcap U$. Closed P-subsets of ω^* are the Stone spaces of algebras $P(\omega)/I$, where I is a p-ideal; and closed nowhere dense subsets of ω^* are Stone spaces of algebras $P(\omega)/I$, where I is tall.

E. K. van Douwen and J. van Mill asked whether one can prove in ZFC the existence of nowhere dense P-subsets of ω^* which are homeomorphic to ω^*. (See [vM, p. 537]. Also, consult this article for information about the situation under CH). This question translates as follows.

Question 1. Can one prove in ZFC that there is a tall p-ideal I such that the algebras $P(\omega)/I$ and $P(\omega)/Fin$ are isomorphic?

We say that an ideal I is *trivial below* a subset $B \subset \omega$, iff there exists an $A \subset B$ so that $I \cap P(B)$ is generated by $(Fin \cap P(B)) \cup \{A\}$. We denote: $\mathrm{Tr}(I) = \{B \subset \omega: I \text{ is trivial below } B\}$.

Received by the editors October 31, 1988 and, in revised form, December 29, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54D40, 06E05; Secondary 03E35.
Key words and phrases. Čech-Stone compactification, $\beta\omega$, embeddings of Boolean algebras.
The preparation of the final version of this paper was partially supported by NSERC grant A-7345.

Claim 2. (a) $I \subseteq \mathrm{Tr}(I)$ for every ideal I.

(b) An ideal I is tall iff $\mathrm{Tr}(I) = I$. $\square$

For any subfamily $A \subseteq P(\omega)/Fin$ denote $[A] = \{A \in \omega \colon A/Fin \in A\}$. Let AKF abbreviate the following statement: "For every homomorphism $\underline{H}$: $P(\omega)/Fin \to P(\omega)/Fin$ and every uncountable family B of pairwise almost disjoint subsets of ω there exists a $B \in \mathrm{B}$ so that $B \in \mathrm{Tr}([\mathrm{Ker}(\underline{H})])$."

In [J], I proved the relative consistency of AKF with ZFC and studied some of its consequences. Here we show that AKF yields an answer to Question 1.

Theorem 3. *Suppose AKF holds, and I is a tall p-ideal so that the algebra $P(\omega)/I$ can be isomorphically embedded into $P(\omega)/Fin$. Then the ideal I is countably saturated, i.e. the quotient algebra $P(\omega)/I$ satisfies the c.c.c.*

Corollary 4. *AKF implies that no nowhere dense P-subset of ω^* is homeomorphic to ω^*.* $\square$

Proof of Theorem 3. Suppose that AKF holds, and that I is a tall p-ideal so that the algebra $P(\omega)/I$ does not satisfy the c.c.c. Let $\langle A_\xi \colon \xi < \omega_1 \rangle$ be a sequence of subsets of ω so that $A_\xi \notin I$ and $A_\xi \cap A_\eta \in I$ for $\xi < \eta < \omega_1$. Let $\eta < \omega_1$. Since I is a p-ideal, there exists a $C_\eta \in I$ so that $A_\xi \cap A_\eta - C_\eta \in Fin$ for all $\xi < \eta$. Let $B_\eta = A_\eta - C_\eta$. Then $B_\eta \notin I$, and $B_\xi \cap B_\eta \in Fin$ for all $\xi < \eta < \omega_1$.

Suppose now towards a contradiction that there is a homomorphism $\underline{H}$: $P(\omega)/Fin \to P(\omega)/Fin$ so that $[\mathrm{Ker}(\underline{H})] = I$. By AKF there is some $\xi < \omega_1$ so that $B_\xi \in \mathrm{Tr}([\mathrm{Ker}(\underline{H})])$. By claim 2(b), this implies that $B_\xi \in I$. A contradiction. $\square$

In Theorem 4, neither the assumption that I is a p-ideal nor the assumption that I is tall can be dropped.

References

[J] W. Just, *A modification of Shelah's oracle-c.c. with applications*, Preprint, University of Toronto, 1988.

[vM] J. van Mill, *An introduction to $\beta\omega$*, in: Handbook of Set-Theoretical Topology, (K. Kunen & J. E. Vaughan eds.), North Holland, 1984, 503–567.

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