NOWHERE DENSE $P$-SUBSETS OF $\omega^*$

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Abstract. It is relatively consistent with ZFC that no nowhere dense $P$-subset of $\omega^*$ is homeomorphic to the space $\omega^*$ itself.

By $P(\omega)$ we denote the Boolean algebra of all subsets of $\omega$. Whenever we use the word 'ideal' we mean a proper ideal $I \subset P(\omega)$ such that $I$ contains $Fin$—the ideal of finite subsets of $\omega$. An ideal $I$ is called a $p$-ideal, if for every countable subset $\{A_n: n \in \omega\} \subset I$ there exists a $B \in I$ such that $A_n - B \in Fin$ for every $n$. An ideal $I$ is called tall, if for every infinite $B \subset \omega$ there exists an infinite $A \in I$ such that $A \subset B$. For instance, $Fin$ is a $p$-ideal, but not tall.

The Stone space of $P(\omega)/Fin$ is $\omega^*$, i.e. $\beta\omega - \omega$. A subset $Y$ of a topological space $X$ is called a $P$-set, if for every countable family $U$ of open supersets of $Y$ there exists an open $V$ so that $Y \subset V \subset \bigcap U$. Closed $P$-subsets of $\omega^*$ are the Stone spaces of algebras $P(\omega)/I$, where $I$ is a $p$-ideal; and closed nowhere dense subsets of $\omega^*$ are Stone spaces of algebras $P(\omega)/I$, where $I$ is tall.

E. K. van Douwen and J. van Mill asked whether one can prove in ZFC the existence of nowhere dense $P$-subsets of $\omega^*$ which are homeomorphic to $\omega^*$. (See [vM, p. 537]. Also, consult this article for information about the situation under CH). This question translates as follows.

Question 1. Can one prove in ZFC that there is a tall $p$-ideal $I$ such that the algebras $P(\omega)/I$ and $P(\omega)/Fin$ are isomorphic?

We say that an ideal $I$ is trivial below a subset $B \subset \omega$, iff there exists an $A \subset B$ so that $I \cap P(B)$ is generated by $(Fin \cap P(B)) \cup \{A\}$. We denote: $Tr(I) = \{B \subset \omega: I$ is trivial below $B\}$. 
Claim 2. (a) $I \subseteq \text{Tr}(I)$ for every ideal $I$.
(b) An ideal $I$ is tall iff $\text{Tr}(I) = I$. □

For any subfamily $A \subseteq P(\omega)/\text{Fin}$ denote $[A] = \{A \in \omega: A/\text{Fin} \in A\}$. Let AKF abbreviate the following statement: “For every homomorphism $H: P(\omega)/\text{Fin} \to P(\omega)/\text{Fin}$ and every uncountable family $B$ of pairwise almost disjoint subsets of $\omega$ there exists a $B \in B$ so that $B \in \text{Tr}([\text{Ker}(H)])$.”

In [J], I proved the relative consistency of AKF with ZFC and studied some of its consequences. Here we show that AKF yields an answer to Question 1.

**Theorem 3.** Suppose AKF holds, and $I$ is a tall $p$-ideal so that the algebra $P(\omega)/I$ can be isomorphically embedded into $P(\omega)/\text{Fin}$. Then the ideal $I$ is countably saturated, i.e. the quotient algebra $P(\omega)/I$ satisfies the c.c.c.

**Corollary 4.** AKF implies that no nowhere dense $P$-subset of $\omega^*$ is homeomorphic to $\omega^*$. □

**Proof of Theorem 3.** Suppose that AKF holds, and that $I$ is a tall $p$-ideal so that the algebra $P(\omega)/I$ does not satisfy the c.c.c. Let $\langle A_\xi : \xi < \omega_1 \rangle$ be a sequence of subsets of $\omega$ so that $A_\xi \notin I$ and $A_\xi \cap A_\eta \in I$ for $\xi < \eta < \omega_1$. Let $\eta < \omega_1$. Since $I$ is a $p$-ideal, there exists a $C_\eta \in I$ so that $A_\xi \cap A_\eta - C_\eta \in \text{Fin}$ for all $\xi < \eta$. Let $B_\eta = A_\eta - C_\eta$. Then $B_\eta \notin I$, and $B_\xi \cap B_\eta \in \text{Fin}$ for all $\xi < \eta < \omega_1$.

Suppose now towards a contradiction that there is a homomorphism $H: P(\omega)/\text{Fin} \to P(\omega)/\text{Fin}$ so that $[\text{Ker}(H)] = I$. By AKF there is some $\xi < \omega_1$ so that $B_\xi \in \text{Tr}([\text{Ker}(H)])$. By claim 2(b), this implies that $B_\xi \in I$. A contradiction. □

In Theorem 4, neither the assumption that $I$ is a $p$-ideal nor the assumption that $I$ is tall can be dropped.

**References**
