A CONSTRUCTION FOR PSEUDOCOMPLEMENTED SEMILATTICES AND TWO APPLICATIONS

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Abstract. A method is given by which pseudocomplemented semilattices can be constructed from graphs. Two consequences of the method are obtained, namely: there exist continuum-many quasivarieties of pseudocomplemented semilattices; for any non-zero cardinal \( \kappa \), there exist \( \kappa \) pairwise non-isomorphic pseudocomplemented semilattices with isomorphic endomorphism monoids.

1. INTRODUCTION

A pseudocomplemented semilattice is an algebra \((S; \wedge, *, 0, 1)\) of type \((2, 1, 0, 0)\) consisting of a semilattice \((S; \wedge)\) with a least element 0, a greatest element 1, and a pseudocomplementation operation \(*\) such that, for \(x, y \in S\), \(x \wedge y = 0\) if and only if \(y \leq x^*\).

We shall exhibit a method by which pseudocomplemented semilattices can be constructed from graphs. By applying this construction, the following results will be established.

Theorem 1. There exist \(2^{\aleph_0}\) quasivarieties of pseudocomplemented semilattices.

Theorem 2. Given a non-zero cardinal \( \kappa \), there exists a family of pseudocomplemented semilattices \((S_i: i \in I)\) satisfying

(i) \(S_i \neq S_j\) for distinct \(i, j \in I\);
(ii) \(\text{End}(S_i) \cong \text{End}(S_j)\) for all \(i, j \in I\);
(iii) if \( \kappa \) is infinite, then \(|I| = 2^\kappa\) and \(|S_i| = \kappa\) for all \(i \in I\);
(iv) if \( \kappa \) is finite, then \(|I| = \kappa\) and each \(S_i\) is finite.

In Theorem 2, and throughout this paper, the notation \(\text{End}(S)\) denotes the monoid (semigroup with identity) of all endomorphisms of \(S\) with composition as multiplication.

It should be noted that Theorem 1 stands in striking contrast to the fact that there are only two non-trivial varieties of pseudocomplemented semilattices.
proved by Jones [7] (cf. Sankappanavar [10]). That not every quasivariety
of pseudocomplemented semilattices is necessarily a variety had already been
observed by Sankappanavar (unpublished) and, independently, Schmid [12].

Both theorems have precise analogues for pseudocomplemented distributive
lattices, i.e., algebras \((L; \vee, \wedge, \ast, 0, 1)\) such that \((L; \vee, \wedge)\) is a distributive
lattice and \((L; \wedge, \ast, 0, 1)\) is a pseudocomplemented semilattice. The analogue
of Theorem 1 is given in [1] and, independently, Wroński [13]; the analogue
of Theorem 2 was proved in [2]. For Boolean algebras, both theorems fail
spectacularly: there is only one non-trivial quasivariety, and Boolean algebras
having isomorphic endomorphism monoids are isomorphic (Magill [8], Maxson
[9], and Schein [11]).

The reader in need of background material concerning pseudocomplemented
semilattices and related topics is directed to Grätzer [3]. We shall use the fol-
nowing notations. If \(S\) is a pseudocomplemented semilattice, \(S^*\) denotes the
skeleton of \(S\), that is, \(S^* = \{x^*: x \in S\}\). The Glivenko congruence on \(S\),
denoted \(\Gamma_S\), is defined by \(\Gamma_S = \{(x, y) \in S \times S: x^* = y^*\}\). The Glivenko endo-
morphism of \(S\), denoted \(\gamma_S\), is defined by \(\gamma_S(x) = x^{**}\) for all \(x \in S\). Because
\(x^* = x^{***}\) for all \(x \in S\), it follows that \(\gamma_S\) is the identity on \(S^*\) and that \(\Gamma_S\)
is the congruence induced by \(\gamma_S\), i.e., \(\Gamma_S = \{(x, y) \in S \times S: \gamma_S(x) = \gamma_S(y)\}\).
Since \(S^*\) is a Boolean lattice where \(x^* \lor y^* = (x^{**} \land y^{**})^*\) for any \(x, y \in S\),
it follows that if \(\varphi: S \rightarrow T\) is a homomorphism to a pseudocomplemented
semilattice \(T\), then \(\varphi \upharpoonright S^*: S^* \rightarrow T^*\) is a Boolean homomorphism.

2. The construction

The immediate goal of this section is to define a pseudocomplemented semi-
lattice \(S_G\) for every graph \(G\).

For a graph \(G = (V, E)\) (i.e., a set \(V\) together with a set \(E\) of two-element
subsets of \(V\)), let \(B_G\) denote the Boolean lattice of finite/co-finite subsets of
\(V\) ordered by inclusion. Further, let

\[
S_G = (B_G \times 2) \setminus \{((\emptyset, 1), (V, 0)) \cup \{(a, 0): |a| = 1 \text{ or } a \in E\}\}
\]

where 2 denotes the two-element chain \(\{0, 1\}\). Let \(\leq\) denote the usual
ordering on \(B_G \times 2\) and define a relation \(\leq\) on \(S_G\) by \((a, i) \leq (b, j)\) iff
either \((a, i) \leq (b, j)\) in \(B_G \times 2\),

or \(i = 1, j = 0, a \leq b, \text{ and } |a| = 1\),
or \(i = 1, j = 0, a \leq b, \text{ and } a \in E\).

It must be shown that \((S_G, \leq)\) is indeed a pseudocomplemented semilattice.

Lemma 1. \(\leq\) is an order relation.

Proof. Since \(\leq\) is reflexive, so too is \(\leq\).

To see that \(\leq\) is anti-symmetric, first note that the only new pairs added to
\(\leq\) are of the form \((a, 1) \leq (b, 0)\), whence we may suppose that

\((a, 1) \leq (b, 0)\) and \((b, 0) \leq (a, 1)\).
It follows that $a = b$. Further, by the first inequality, either $|a| = 1$ or $a \in E$ which is absurd, since in neither case is $(a, 0)$ an element of $S_G$.

Finally, to see that $\leq$ is transitive, we need only consider cases in which one inequality is of the form $(a, 1) \leq (b, 0)$. Thus, either

$$(c, i) \leq (a, 1) \quad \text{and} \quad (a, 1) \leq (b, 0) \quad \text{or} \quad (a, 1) \leq (b, 0) \quad \text{and} \quad (b, 0) \leq (c, i)$$

for $i = 0, 1$. In either case, $a \leq b$ and $|a| = 1$ or $a \in E$. If, in the former case, $c = \emptyset$, then $i = 0$ and $(c, 0) \leq (b, 0)$. Otherwise, since $c \leq a \leq b$, $|c| = 1$ or $c = a \in E$. No matter which, $i = 1$ and $(c, 1) \leq (b, 0)$. In the latter case $a \leq c$ and, hence, $(a, 1) \leq (c, i)$. \qed

**Lemma 2.** $S_G$ is a semilattice where, for $(a, i), (b, j) \in S_G$,

$$(a, i) \land (b, j) = \begin{cases} 
(\emptyset, 0) & \text{if } a \land b = \emptyset,
(a \land b, 1) & \text{if } |a \land b| = 1 \text{ or } a \land b \in E,
(a \land b, i \land j) & \text{otherwise}.
\end{cases}$$

**Proof.** Suppose $(a, i), (b, j) \in S_G$ and $(c, k)$ is a common lower bound. In particular, $c \leq a \land b$. We consider the various possibilities.

If $a \land b = \emptyset$, then $(\emptyset, 0)$ is a common lower bound. Since $c = \emptyset$ in this case, $(\emptyset, 0)$ is the only lower bound.

If $|a \land b| = 1$ or $a \land b \in E$, then $(a \land b, 1)$ is a common lower bound. Moreover, it is the greatest since $c \leq a \land b$ implies that $(c, k) \leq (a \land b, 1)$ for any $k$.

Finally, suppose $|a \land b| \geq 2$ and $a \land b \notin E$. If $(a \land b, i \land j) = (V, 0)$, then either $(a, i)$ or $(b, j) = (V, 0)$ which is absurd. Thus, $(a \land b, i \land j) \in S_G$ by hypothesis. Clearly, $(a \land b, i \land j)$ is a common lower bound; it is to be seen that it is the greatest. If $k = 0$, then $(c, 0) \leq (a \land b, i \land j)$ automatically. Suppose, on the other hand, that $k = 1$. For $i = j = 1$, $(c, 1) \leq (a \land b, 1)$. Otherwise, say, $i = 0$. Thus $(c, 1) \leq (a, 0)$ and so $|c| = 1$ or $c \in E$. Either way, $(c, 1) \leq (a \land b, i \land j)$. \qed

**Lemma 3.** $S_G$ is a pseudocomplemented semilattice where $(V, 1)^* = (\emptyset, 0)$ and, for $(V, 1) \neq (a, i) \in S_G, (a, i)^* = (a^*, 1)$.

**Proof.** Obviously, $(V, 1)^* = (\emptyset, 0)$. For $(V, 1) \neq (a, i) \in S_G$, $a \neq V$. Thus, $a^* \neq \emptyset$ and, hence, $(a^*, 1) \in S_G$. By Lemma 2, $(a, i) \land (a^*, 1) = (\emptyset, 0)$. Furthermore, if $(a, i) \land (b, j) = (\emptyset, 0)$, then $a \land b = \emptyset$. In particular, $b \leq a^*$ and $(b, j) \leq (a^*, 1)$. \qed

The remainder of this section establishes the properties of pseudocomplemented semilattices of the form $S_G$ that will be required in the proofs of Theorems 1 and 2.

Let $G = (V, E)$ and $H = (W, F)$ be two graphs such that $|V| \geq 5$ and, for every $x \in V$, $x \in e$ for some $e \in E$. Further, let $\phi: S_G \to S_H$ be a homomorphism and let $\theta$ denote the congruence on $S_G$ induced by $\phi$. 

Lemma 4. If, for every co-atom $a$ of $B_G$, $(a, 0) \equiv (a, 1)(\theta)$, then $\theta \supseteq \Gamma_{S_G}$.

Proof. For any $(b, 0), (b, 1) \in S_G$, there exists a co-atom $a \in B_G$ such that $a > b$. Since $(b, 0) \in S_G$, $|b| \neq 1$ and $b \notin E$ and, since $(b, 1) \in S_G$, $b \neq \emptyset$. Thus, by Lemma 2, $(a, 0) \wedge (b, 1) = (b, 0)$ and $(a, 1) \wedge (b, 1) = (b, 1)$.

By hypothesis, it follows that $(b, 0) \equiv (b, 1)(\theta)$.

Lemma 5. If $\theta \nsubseteq \Gamma_{S_G}$, then $\phi$ is one-to-one on $S_G^\ast$.

Proof. Suppose $\phi$ is not one-to-one on $S_G^\ast$. Then, since $\phi : S_G^\ast \rightarrow S_H^\ast$ is a Boolean homomorphism, $\phi(a, 1) = \phi(V, 1) = (W, 1)$ for some co-atom $a \in B_G$. Thus, by Lemma 3, $(\phi(a, 0))^* = \phi((a, 0)^*) = \phi(a^*, 1) = \phi((a, 1)^*) = (\phi(a, 1))^* = (W, 1)^* = (\emptyset, 0)$. It follows that $\phi(a, 0) = (W, 1)$ and, hence, $(a, 0) \equiv (V, 1)(\theta)$. Let $b$ be any other co-atom of $B_G$. By Lemma 4, it is sufficient to show $(b, 0) \equiv (b, 1)(\theta)$. Since $|V| \geq 5$, $|a \wedge b| \geq 3$. Thus, by Lemma 2, $(a \wedge b, 0) = (a, 0) \wedge (b, 1) \equiv (V, 1) \wedge (b, 1) = (b, 1)$. It follows that $(b, 0) \equiv (b, 1)(\theta)$ since $(a \wedge b, 0) \equiv (b, 0) \equiv (b, 1)$.

Lemma 6. If $\theta \nsubseteq \Gamma_{S_G}$, then, for $\emptyset \neq a \in B_G$, $\phi(a, 1) = (r, 1)$ for some $r \in B_H$.

Furthermore,

(i) if $|a| = 1$, then $|r| = 1$;

(ii) if $a \in E$, then $r \in F$; and

(iii) if $(a, 0) \equiv (a, 1)(\theta)$, then $|a| = 2$ and $r \in F$.

Proof. By Lemma 5, $\phi$ is one-to-one on $S_G^\ast$. Further, since $\phi : S_G^\ast \rightarrow S_H^\ast$, it follows that, for $\emptyset \neq a \in B_G$, $\phi(a, 1) = (r, 1)$ for some $r \in B_H$.

With (iii) in mind, suppose $(a, 0) \equiv (a, 1)(\theta)$ for some $|a| \geq 3$. Then there exists a co-atom $b \in B_G$ such that $b \geq a$. Since $\phi$ is order-preserving, $s \geq r$ where $\phi(b, 1) = (s, 1)$. By hypothesis, $(a, 1) \wedge (b, 0) = (a, 0)$ and, hence, $(r, 1) \wedge \phi(b, 0) = (r, 1)$; in particular, $\phi(b, 0) \equiv (r, 1)$. However, since $\phi$ preserves $\ast$, $\phi(b, 0) \in \{(s, 0), (s, 1)\}$. But, because $\phi$ is one-to-one on $S_G^\ast$, $|r| \geq 3$ and, hence, $(s, 0) \notin (r, 1)$. It follows that $\phi(b, 0) = (s, 1)$ and, consequently, that $(b, 0) \equiv (b, 1)(\theta)$. We claim that, for any co-atom $c \in B_G$, $(c, 0) \equiv (c, 1)(\theta)$. To see this observe that, since $|b \wedge c| \geq 3$, $(b \wedge c, 0) = (b, 0) \wedge (b \wedge c, 1) = (b, 1) \wedge (b \wedge c, 1) = (b \wedge c, 1)$. Hence, $\phi(b \wedge c, 1) = \phi(b \wedge c, 0) = \phi((c, 0) \wedge (b \wedge c, 1)) = \phi(b \wedge c, 1)$; that is, $\phi(c, 0) \equiv \phi(b \wedge c, 1)$. If $\phi(c, 1) = (t, 1)$, then, since $\phi$ preserves $\ast$, $\phi(c, 0) \in \{(t, 0), (t, 1)\}$. But, if $\phi(b \wedge c, 1) = (u, 1)$, then $|u| \geq 3$ since $\phi$ is one-to-one on $S_G^\ast$. It follows that $(t, 0) \notin (u, 1)$ and, hence, $\phi(c, 0) = (t, 1)$. Thus, $(c, 0) \equiv (c, 1)(\theta)$ for every co-atom $c \in B_G$. By Lemma 4, this is absurd. Thus, we conclude that, for any $a \in B_G$, if $|a| \geq 3$, then $(a, 0) \neq (a, 1)(\theta)$.

Suppose now that $|a| = 2$. Set $b = a \cup \{x\}$ for some $x \in V \setminus a$. Since $\phi$ is one-to-one on $S_G^\ast$, $|s| > |r| \geq 2$ where $\phi(a, 1) = (r, 1)$ and $\phi(b, 1) = (s, 1)$.

Thus, as shown above, $(b, 0) \equiv (b, 1)(\theta)$ and, in particular, $\phi(b, 0) = (s, 0)$. To prove (ii), suppose $a \in E$. Then $(a, 1) \equiv (b, 0)$ and, hence, $(r, 1) \equiv (s, 0)$ which, since $|r| \geq 2$, implies $r \in F$. Thus, (ii) is seen to hold. To prove
(iii), suppose \( a \not\in E \). For \( r \not\in F \), since \( (a, 0) = (a, 1) \land (b, 0) \), \( \varphi(a, 0) = (r, 1) \land (s, 0) = (r, 0) \) and, hence, \( (a, 0) \not= (a, 1) \theta \) which verifies (iii).

Since \( \varphi \) is one-to-one on \( S^*_G \) and \( V \subseteq \bigcup E \), (i) is an immediate consequence of (ii). \( \square \)

**Proposition 1.** If \( \theta \not\in \Gamma_{S_0} \), then the following hold:

(i) for \( x \in V \), \( \varphi({x}, 1) = (\{\psi({x})\}, 1) \) defines a one-to-one compatible mapping \( \psi: G \to H \) (a mapping is compatible if \( \psi(x), \psi(y) \in F \) whenever \( \{x, y\} \in E \) which is also onto whenever \( G \) is finite;

(ii) if \( G = H \) and \( \varphi \upharpoonright S^*_G \) is the identity, then \( \varphi \) is the identity.

**Proof.** By Lemma 5, \( \varphi \) is one-to-one on \( S^*_G \). Thus, by Lemma 6 (i) and (ii), \( \psi: G \to H \) as given above is a well-defined one-to-one compatible mapping.

To complete the proof of (i), suppose \( G \) is finite. Then, since \( \bigwedge({x}, 1): x \in V \) = (\( \emptyset, 0 \)), it follows that

\[
\bigwedge((\{\psi({x})\}, 1): x \in V) = \bigwedge((\{\psi({x})\}, 1)^*: x \in V) = \bigwedge((\varphi(\{x\}), 1)^*: x \in V) = \bigwedge((\varphi(\{x\}), 1): x \in V)
\]

But as a meet of co-atoms of \( S_H \), \( \bigwedge((\{\psi({x})\}, 1): x \in V) \) can be \( (\emptyset, 0) \) only if every co-atom is present, that is, \( \psi(V) \supseteq W \).

Finally, if \( G = H \) and \( \varphi \upharpoonright S^*_G \) is the identity, then, by Lemma 6 (iii), \( \varphi \) is the identity and (ii) holds. \( \square \)

### 3. Proof of Theorem 1

For \( i < \aleph_0 \), let \( G(i) = (V(i), E(i)) \) be the complete graph on \( 5 + i \) elements, and let \( S_{G(i)} \) be the associated pseudocomplemented semilattice. Clearly, for \( i < \aleph_0 \), \( |V(i)| \geq 5 \) and \( V(i) \subseteq \bigcup E(i) \).

Let \( (U_\alpha: \alpha < 2^{\aleph_0}) \) be a family of \( 2^{\aleph_0} \) distinct subsets of \( \aleph_0 \), and for each \( \alpha < 2^{\aleph_0} \) let \( Q_\alpha \) denote the quasivariety generated by the set \( \{S_{G(i)}: i \in U_\alpha\} \).

Fix \( \alpha, \beta < 2^{\aleph_0} \) with \( \alpha \not= \beta \). Without loss of generality, we may choose \( m \in U_\alpha \setminus U_\beta \). If \( S_{G(m)} \in Q_\beta \), then \( S_{G(m)} \in S_{G(i): i \in U_\beta} \) (see Grätzer and Lakser [4]), and it follows (see Grätzer, Lakser, and Quackenbush [5]) from the fact that pseudocomplemented semilattices are locally finite (see Jones [7] and also Sankappanavar [10]) that \( S_{G(m)} \in \text{SP}(S_{G(i): i \in U_\beta}) \). Given any \( x \in S_{G(m)} \) with \( x \not= x^{**} \), it follows that there exists \( i \in U_\beta \) and a homomorphism \( \varphi: S_{G(m)} \to S_{G(i)} \) such that \( \varphi(x) \not= \varphi(x^{**}) \). Hence the congruence induced by \( \varphi \) fails to contain the Glivenko congruence. By Proposition 1 (i) it follows that \( |G(m)| = |G(i)| \), which is absurd, and so the quasivarieties \( (Q_\alpha: \alpha < 2^{\aleph_0}) \) are distinct.
4. Proof of Theorem 2

To establish Theorem 2, we again choose suitable families of graphs, but this time with a little more care. By Hedrlín and Sichler [6], given a non-zero cardinal $\kappa$, there exists a family of graphs $(G(i) = (V(i), E(i)) : i \in I)$ with the following properties: (i) each $G(i)$ is rigid, i.e., the only compatible mapping from $G(i)$ to itself is the identity; (ii) for $i \neq j$, there is no compatible mapping from $G(i)$ to $G(j)$; (iii) if $\kappa$ is infinite, then $|I| = 2^\kappa$ and $|V(i)| = \kappa$ for $i \in I$; (iv) if $\kappa$ is finite, then $|I| = \kappa$ and $5 \leq |V(i)| = |V(j)| < \kappa_0$ for $i, j \in I$.

Consider $(SG(i), \gamma_i : i \in I)$. By Proposition 1 (i), the absence of a compatible map from $G(i)$ to $G(j)$ implies that $S_{G(i)} \not\cong S_{G(j)}$ for $i \neq j$.

For $i \in I$, let $\gamma_i$ be the Glivenko endomorphism of $S_{G(i)}$.

We show that for $i \in I$, $\text{End}(S_{G(i)})$ is isomorphic to the monoid $M \cup \{1\}$ where $M = \text{End}(B_{G(i)})$ and 1 is an adjoined identity element, that is, $1 \notin M$, $i^2 = 1$, and $i \psi = \psi i = \psi$ for all $\psi \in M$. Since $B_{G(i)} \cong B_{G(j)}$ for all $i, j \in I$, it will follow that $\text{End}(S_{G(i)}) \cong \text{End}(S_{G(j)})$.

As $B_{G(i)} \cong S_{G(i)}^*$ it suffices to establish an embedding $T : \text{End}(S_{G(i)}^*) \to \text{End}(S_{G(i)})$ such that the image of $T$ consists precisely of all non-identity endomorphisms of $S_{G(i)}$. Define $T$ by $T(\alpha) = \alpha \gamma_i$ for all $\alpha \in \text{End}(S_{G(i)}^*)$.

Since $\gamma_i$ is the identity on $S_{G(i)}$, $T$ is one-to-one and a homomorphism. Moreover, all $T(\alpha)$ are non-identity because $\gamma_i$ is not one-to-one.

To show that every non-identity $\phi \in \text{End}(S_{G(i)})$ belongs to the image of $T$, it will suffice to show that each such $\phi$ is equal to $\phi' \gamma_i$, where $\phi'$ denotes the restriction of $\phi$ to $S_{G(i)}^*$. Equivalently, we need to show that the congruence induced by $\phi$ contains the Glivenko congruence on $S_{G(i)}$. If such is not the case, then, by Proposition 1 (i), the rigidity of $G(i)$ implies that $\phi'$ is the identity. Whence, by Proposition 1 (ii), $\phi$ is the identity in violation of the hypothesis.

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