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MATROIDS DETERMINE THE EMBEDDABILITY OF GRAPHS IN SURFACES

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ABSTRACT. The embeddability of a graph in a given surface is determined entirely by the polygon matroid of the graph. That is also true for cellular embeddability in nonorientable surfaces but not in orientable surfaces.

An *embedding* of a finite graph Γ in a surface S is a homeomorphism of Γ, regarded as a topological space, with a closed subset of S. In order to know in which surfaces Γ embeds it suffices to consider only the compact surfaces: the orientable ones T_g of genus g (Euler characteristic $2-2g$) for $g \geq 0$, and the nonorientable ones U_h of Euler characteristic $2-h$ for $h \geq 1$. For uniformity of terminology we define the *demigenus* d of a compact surface by $d(T_g) = 2g$, $d(U_h) = h$. One knows exactly which compact surfaces can embed Γ if one knows two parameters: the *genus* of the graph, $g(\Gamma) = \min\{g : \Gamma$ embeds in $T_g\}$, and its *crosscap number* (also called *nonorientable genus*) $h(\Gamma) = \min\{h : \Gamma$ embeds in $U_h\}$. A natural companion to these is the *demigenus of* Γ (also known as *generalized genus*, *Euler genus*, etc.),

$$d(\Gamma) = \min\{2g(\Gamma), h(\Gamma)\},$$

the smallest demigenus of a compact surface in which Γ embeds. It is the purpose of this note to point out the apparently unrecognized fact that these three parameters are matroidal, that is, determined by the polygon matroid [1] of the graph. This fact, which generalizes Whitney's theorem that planarity is matroidally determined [20], follows readily from published work on graph embedding. [2]

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[1] Also called the 'cycle matroid'. Its circuits are the circuits of the graph. Cf. [17, §1.10] or [18, §6.1].

[2] A general reference for graph embedding is [8]. For matroids see [17] or [18, 19].

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Three operations on a graph Γ are (a) identifying two vertices in different components, (b) the reverse, and (c) *twisting*. The last named consists in splitting Γ into subgraphs Γ_1 and Γ_2 whose intersection is precisely a 2-separating vertex set $\{v, w\}$ such that v and w are connected by a path in Γ_1 and Γ_2, and reconnecting all the edges of Γ_2 at v and w to the opposite vertex, respectively w or v. Plainly, none of these operations changes the polygon matroid $G(\Gamma)$. Whitney's 2-isomorphism theorem [21; 17, §6.1; 18, §6.3] states that, if $G(\Gamma) = G(\Gamma')$, then Γ' can be obtained from Γ by iterating operations (a, b, c). Thus we need to show that these operations do not alter the genus, demigenus, or crosscap number.

The BHKY theorem [2], that $g(\Gamma) = \sum_1^n g(\Gamma_i)$ where $\Gamma_1, \dots, \Gamma_n$ are the blocks of Γ, shows that genus is unaffected by (a) and (b). The observation of [15, Corollary 2] that the analogous formula holds for the demigenus of a connected graph implies a similar conclusion for $d(\Gamma)$ if Γ is connected. A simple argument from [15] gives the crosscap number as well: For any graph let $\delta(\Gamma) = h(\Gamma) - d(\Gamma)$. As noted in [15, Eq. (1)], $\delta(\Gamma) = 0$ or 1. Now let Γ have blocks $\Gamma_1, \dots, \Gamma_n$. If Γ is connected,

$$h(\Gamma) = d(\Gamma) + \delta(\Gamma) = \sum_{i=1}^{n} d(\Gamma_i) + \delta(\Gamma). \tag{1}$$

[15, Theorem 1] states that

$$\delta(\Gamma) = 1 \quad \Leftrightarrow \quad \text{all } \delta(\Gamma_i) = 1. \tag{2}$$

Thus $h(\Gamma)$ is determined by the blocks of Γ.

If Γ has $k > 1$ components we use a trick of [2, p. 567]. Let Γ' be Γ with $k-1$ edges added to make a connected graph. In Γ' the blocks are $\Gamma_1, \dots, \Gamma_n$ and single edges $\Gamma_{n+1}, \dots, \Gamma_{n+k-1}$. The latter have genus and demigenus 0 and $h(\Gamma_i) = \delta(\Gamma_i) = 1$. Since the genus, demigenus, and crosscap number of Γ' are independent of the location of the extra edges, it is easy to see that $g(\Gamma') = g(\Gamma)$ and $h(\Gamma') = h(\Gamma)$, whence $d(\Gamma') = d(\Gamma)$ and $\delta(\Gamma') = \delta(\Gamma)$. It follows that $d(\Gamma) = \sum_1^n d(\Gamma_i)$ and that (1) and (2) hold for Γ. Therefore $h(\Gamma)$ is determined by the blocks of Γ.

Now suppose $\Gamma = \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 are connected[3] and $\Gamma_1 \cap \Gamma_2$ consists of just the two vertices v and w. The main theorem of Decker et al. [3; 4, Theorem 0.1] is that there is a function $\mu(\Gamma, \{v, w\})$ of connected graphs with a distinguished vertex pair such that

$$g(\Gamma) = g(\Gamma_1) + g(\Gamma_2) + \lceil \tfrac{1}{4}(3 - \mu(\Gamma_1, \{v, w\})\mu(\Gamma_2, \{v, w\})) \rceil.$$

If Δ is a graph and v, $w \in V(\Delta)$, let Δ^{vw} be Δ with an extra edge vw adjoined. Richter [10] proves that

$$d(\Gamma) = \min\{d(\Gamma_1^{vw}) + d(\Gamma_2^{vw}), d(\Gamma_1) + d(\Gamma_2) + 2\}.$$

[3] The requirement of connectedness is not stated in [4] but it is necessary for the proof. The formula may be false if Γ_1 or Γ_2 does not contain a vw path.

In [9] he shows by a more complicated argument that there is a function μ of pairs of connected graphs with a distinguished vertex pair (this μ is unrelated to that of Decker et al.) such that

$$h(\Gamma) = h(\Gamma_1) + h(\Gamma_2) + \mu((\Gamma_1, \{v, w\}), (\Gamma_2, \{v, w\})).$$

These three formulas, together with additivity on blocks, imply that $g(\Gamma)$, $d(\Gamma)$, and $h(\Gamma)$ are invariant under twisting. Hence our main result:

Theorem. *The genus, demigenus, and crosscap number of a graph are determined by its polygon matroid.*

By a *minor* of a graph or matroid B we mean any isomorph of a contraction of a subgraph or submatroid of B. The relation defined by $A \leq B$ if A is a minor of B is a partial ordering of isomorphism types of graphs and also of matroids; we call it the *minor ordering*. It is easy to see that for each surface S the property of embeddability in S is *hereditary*, that is, if Γ embeds so does every minor. Consequently there is a set $\mathscr{F}_G(S)$ of graphs (actually, isomorphism types of graphs) such that Γ is embeddable in S if and only if no minor of Γ belongs to $\mathscr{F}_G(S)$. The members of $\mathscr{F}_G(S)$ are known as the *forbidden graph minors* for embedding in S. Our theorem implies:

Corollary 1. *A graph Γ embeds in S if and only if $G(\Gamma)$ has no minor in the set $\mathscr{F}_M(S) = \{G(F) : F \in \mathscr{F}_G(S)\}$.*

Corollary 2. *A matroid M is the matroid of a graph embeddable in S if and only if it is graphic and has no minor belonging to $\mathscr{F}_M(S)$.*

In other words, the class of matroids whose graphs are embeddable in a given surface is determined by forbidden matroid minors (since the property of graphicity is so determined, according to the famous theorem of Tutte [16]; the five forbidden minors are described in [17, §10.5] and [19, §2.6]). By [12] (see also [11]), and when $S = U_h$ also by [1], $\mathscr{F}_G(S)$ is finite. Consequently the forbidden minors for a matroid to be the polygon matroid of an S-embeddable graph are finite in number.

One might hope that $\mathscr{F}_M(S)$ would be much smaller than $\mathscr{F}_G(S)$, which is very large if $d(S) \geq 2$. But this is not the case for $S = T_0$ or U_1, as one can see by inspection of the two forbidden graph minors for T_0 (i.e., K_5 and $K_{3,3}$) and the 35 for U_1 (they are the first 35 irreducible graphs listed in [7]).

Corollaries 1 and 2 remain true if S is replaced by a pair of surfaces T_g and U_h and embeddability is interpreted as being embeddable in both, or in either, of the surfaces. In either case the forbidden graph minors are finitely many, by [13] and in the second case by [1], hence so are the forbidden matroid minors.

A natural follow-up queston is whether *cellular* embeddability of a graph in a given surface, where every component of the graph's complement in S is an open 2-cell, is a matroidal property. To avoid triviality assume Γ is a connected graph. The *genus range* is $\mathfrak{g}(\Gamma) = \{g : \Gamma$ has a cellular embedding in $T_g\}$ and the *crosscap range* is $\mathfrak{h}(\Gamma) = \{h : \Gamma$ embeds cellularly in $U_h\}$. Each of these sets is finite and, if nonempty, is *contiguous*: if $i < k < j$ and i, j are in the set, so is k. (See [5, Theorem 3.2] and [14, Theorem 8], or consult [8, §3.4].) Obviously $g(\Gamma) = \min \mathfrak{g}(\Gamma)$; we define $g_{\max}(\Gamma) = \max \mathfrak{g}(\Gamma)$. It is clear that $\mathfrak{h}(\Gamma) = \varnothing$ if Γ is a tree and otherwise $h(\Gamma) = \min \mathfrak{h}(\Gamma)$; we set $h_{\max}(\Gamma) = \max \mathfrak{h}(\Gamma)$. From work of Edmonds [6] it follows directly (see [8, Theorem 3.4.3]) that $h_{\max}(\Gamma) = \beta_1(\Gamma)$, the cyclomatic number of Γ. This is precisely the nullity of $G(\Gamma)$. Thus the crosscap range is matroidal.

A theorem of Xuong ([22]; see [8, Corollary to Theorem 3.4.13]) says that $g_{\max}(\Gamma) = \frac{1}{2}(\beta_1(\Gamma) - \xi(\Gamma))$, where $\xi(\Gamma)$ is the minimum over all spanning trees T of the number of components of $\Gamma \backslash E(T)$ which have an odd number of edges. This quantity is unfortunately not determined by $G(\Gamma)$. For example let Γ_1 and Γ_2 be formed from the three blocks K_3, K_3, K_2. To construct Γ_1 we join each K_3 to a different vertex of the K_2. Obviously $\xi(\Gamma_1) = 2$. To form Γ_2 we join the two K_3's at a vertex and attach the K_2 anywhere. Evidently $\xi(\Gamma_2) = 0$. Yet the two graphs have the same matroid because they have the same blocks.

To summarize:

Proposition. *The crosscap range of a connected graph is determined by its matroid, but the genus range is not.*

One wonders how much information about the genus range is lost by passing to the matroid. Let $g_{\max}(M)$, for a graphic matroid M, be

$$\max\{g_{\max}(\Gamma) : G(\Gamma) = M\}.$$

Is $g_{\max}(M) - g_{\max}(\Gamma)$ for graphs with $G(\Gamma) = M$ bounded by a constant, or by a small multiple of $g_{\max}(M)$?

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