ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO 
\( \Delta u + Ku^\sigma = 0 \) ON \( \mathbb{R}^n \) FOR \( n \geq 3 \)

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Abstract. The equation \( \Delta u + Ku^\sigma = 0 \) is considered in \( \mathbb{R}^n \) for \( n \geq 3 \), \( K \) a Hölder continuous function and \( \sigma \) a positive constant. If \( K = O(|x|^{-l}) \) for \( l > 2 \), we determine the asymptotic behavior of bounded solutions. In the case \( K \) is nonpositive and \( \sigma \) is greater than one, we show that the first term in the asymptotic description may be chosen arbitrarily.

1. Introduction

Let \( K \) be a Hölder continuous function, not identically zero, on \( \mathbb{R}^n \), \( n \geq 3 \), and \( \sigma \) a positive constant greater than 1. Ni [9] has shown that if \( K = O(|x|^{-l}) \) as \(|x| \to \infty \) for \( l > 2 \), then the equation

\( \Delta u + Ku^\sigma = 0 \)

has an infinite number of bounded positive solutions that are bounded away from 0.

For \( \Delta u + Ku^\sigma = 0 \) on all of \( \mathbb{R}^n \), Ni [9] shows that for \( l > 2 \), if \( K \) doesn’t change sign then a bounded solution \( u \) approaches a constant \( u_\infty \) as \(|x| \to \infty \). Naito [8] shows that the sign condition on \( K \) is unnecessary and that the constant can be specified arbitrarily in an interval determined by \( K \). In [5], Li and Ni show that the next term in the asymptotic description of \( u \) is given by

\[
|u - u_\infty| = \begin{cases} 
O(|x|^{2-n}) & \text{if } l > n; \\
O(|x|^{2-n} \log |x|) & \text{if } l = n; \\
O(|x|^{2-l}) & \text{if } 2 < l < n.
\end{cases}
\]  

The first result in this paper gives a more complete asymptotic description of solutions to \( \Delta u + Ku^\sigma = 0 \) on \( \mathbb{R}^n \) in the case that \( l > n \). Here we may take any \( \sigma > 0 \).

Theorem 1. Suppose \( K = O(|x|^{-l}) \) as \(|x| \to \infty \) for some \( l > 2 \) and \( \sigma > 0 \). If \( u \) is a bounded positive solution to \( \Delta u + Ku^\sigma = 0 \) then

(i) there is a constant \( u_\infty \) such that \( u - u_\infty = O(|x|^\gamma) \) as \(|x| \to \infty \) for all \( \gamma > \max\{2-l,2-n\} \)
(ii) if \( l > n + m \) then there are \( m + 1 \) homogeneous harmonic polynomials \( f_0, f_1, \ldots, f_m \) where \( f_j \) is of degree \( j \) such that
\[
 u - u_\infty - \frac{f_0}{|x|^{n-2}} - \cdots - \frac{f_m}{|x|^{n-2+2m}} = O(|x|^{\gamma}) \quad \text{as} \quad |x| \to \infty
\]
for all \( \gamma > \max\{2 - l, 1 - n - m\} \).

Therefore, if \( K \) has compact support or decays exponentially as \( |x| \to \infty \) we get a full asymptotic expansion of \( u \).

In the case \( K \leq 0 \), we improve Naito’s result to show that the first term in the asymptotic description can be chosen arbitrarily.

**Theorem 2.** Suppose \( K = O(|x|^{-l}) \) with \( l > 2 \) and \( K \leq 0 \) on \( \mathbb{R}^n \) and \( \sigma > 1 \). Given any \( c > 0 \), there exists a unique positive solution \( u \) to \( \Delta u + Ku^\sigma = 0 \) with \( u - c = O(|x|^{\gamma}) \) as \( |x| \to \infty \) for all \( \gamma > \max\{2 - l, 2 - n\} \).

The techniques used to prove both theorems involve knowledge of how the Laplacian acts on weighted Sobolev spaces as discussed in the next section.

**Remarks.**
1. All of the above results require \( l > 2 \). In the case \( K \leq 0 \) and \( |K| \geq C|x|^{-l} \) as \( |x| \to \infty \) for some \( l \leq 2 \), Ni shows that there are no positive solutions. Li and Ni [5] have some results for \( K \geq 0 \) and \( 0 < l < 2 \) for the radial case.

2. It is interesting to compare the asymptotics (B) with the results of Meyers in [7]. Meyers studies the Poisson equation on an exterior domain. He has shown that if \( \Delta u = f \) where \( f = O(|x|^{-l}) \) as \( |x| \to \infty \) then there is a solution \( u \) with
\[
 u = \begin{cases} 
 O(|x|^{2-l}) & \text{if } 2 < l < n \text{ or } l \text{ is not an integer;} \\
 O(|x|^{2-l} \log |x|) & \text{otherwise.}
\end{cases}
\]

2. Preliminaries

The weighted space \( W^p_{s,\delta} \), for \( 1 < p < \infty \), \( s \) a nonnegative integer and \( \delta \) real, is defined to be the closure of \( C_0^\infty (\mathbb{R}^n) \) in the norm
\[
 \|u\|_{p,s,\delta} = \sum_{|\alpha| \leq s} \|(1 + |x|^\delta)^{\frac{1}{2}} \partial^\alpha u\|_{L^p}.
\]
If \( s = 0 \) then \( W^p_{0,\delta} = L^p_\delta \).

We use the following results about these spaces. These results can be found in McOwen [6], Cantor [2], and Nirenberg and Walker [10].

1. \( W^p_{s,\delta} \subset W^p_{s',\delta'} \) if \( s \geq s' \) and \( \delta \geq \delta' \). The injection is compact if \( s > s' \) and \( \delta > \delta' \).

2. For \( s > n/p \), if \( f \in W^p_{s,\delta} \) then \( f = O(|x|^{\gamma}) \) as \( |x| \to \infty \) for all \( \gamma > -n/p - \delta \).
(3) If $-n/p < \delta < -n/p + (n-2)$ then there is a constant $A$ such that

$$\|u\|_{p,2,\delta} \leq A \|\Delta u\|_{L^p_{\delta+2}}.$$  

(4) $W^p_{s,\delta} \subset \{\text{continuous functions that approach} \ 0 \ \text{as} \ |x| \to \infty\}$ provided $s > n/p$ and $\delta > -n/p$.

(5) $\Delta: W^p_{2,\delta} \to L^p_{\delta+2}$ is an isomorphism for $-n/p < \delta < (n-2) - n/p$.

(6) $\Delta: W^p_{2,\delta} \to L^p_{\delta+2}$ is a surjection for $-n/p - m - 1 < \delta < -n/p - m$ with nullspace

$$N_m = \bigcup_{j=0}^m \mathcal{H}_j$$

where $\mathcal{H}_j = \{\text{homogeneous harmonic polynomials of degree} \ j\}$. 

(7) $\Delta: W^p_{2,\delta} \to L^p_{\delta+2}$ is an injection for $(n-2) - n/p + m < \delta < (n-2) - n/p + m + 1$ with range

$$\mathcal{R} = \left\{ g \in L^p_{\delta+2} : \int_{R^n} g(y)H(y)dy = 0 \quad \text{for all} \quad H \in \bigcup_{j=0}^m \mathcal{H}_j \right\}.$$  

3. Proof of Theorem 1

Suppose $u$ is a bounded positive solution to $\Delta u + Ku^\sigma = 0$. Choose $p > n$. Let $f = -Ku^\sigma$. Since $u$ is bounded, $f \in L^p_{\delta+2}$ for all $\delta < (l-2) - n/p$. Since $l > 2$, there is a $\delta$ satisfying

$$\frac{-n}{p} < \delta < \min\{(l-2) - \frac{n}{p}, (n-2) - \frac{n}{p}\}.$$  

For such a $\delta$, $\Delta: W^p_{2,\delta} \to L^p_{\delta+2}$ is an isomorphism. Since $f \in L^p_{\delta+2}$, there is $v \in W^p_{2,\delta}$ with $\Delta v = f$. By (4), $v$ is continuous and approaches 0 as $|x| \to \infty$. Since $v \in W^p_{2,\delta}$, $v = O(|x|^\gamma)$ for all $\gamma > -n/p - \delta$. This is true for any choice of $\delta$ with $-n/p < \delta < \min\{(l-2) - n/p, (n-2) - n/p\}$ which means that $v = O(|x|^\gamma)$ for all $\gamma > \max\{2-l, 2-n\}$.

Now we have $v$ continuous and bounded on $R^n$ and $\Delta v = \Delta u = -Ku^\sigma$. Therefore $u - v$ is a bounded harmonic function on $R^n$ so $u - v = u_\infty$, a constant. We get $u - u_\infty = v$ so $u - u_\infty = O(|x|^\gamma)$ for all $\gamma > \max\{2-l, 2-n\}$.

We get more terms in the asymptotic expansion of $u$, with the number of terms depending on how large $l$ is. If $l > n$, we look for a fundamental solution term; that is, a term of the form $f_0/|x|^{n-2}$ where $f_0$ is a constant.

If $l > n$, then we have $u - u_\infty = O(|x|^\gamma)$ for all $\gamma > 2 - n$. We wish to determine the constant $f_0$ so that

$$u \sim u_\infty + \frac{f_0}{|x|^{n-2}} \quad \text{as} \quad |x| \to \infty.$$  

Let $u_0$ be a $C^\infty(R^n)$ function satisfying

$$u_0(x) = \frac{f_0}{|x|^{n-2}} \quad \text{for} \quad |x| \geq 1.$$
where $f_0$ is a constant we wish to determine. We want to have $u - u_\infty - u_0 = O(|x|^{\gamma})$ as $|x| \to \infty$ for all $\gamma > \max\{2 - l, 1 - n\}$. That implies $u - u_\infty - u_0 \in W^{p,\delta}_{2,\delta}$ for all $\delta < \min\{(l - 2 - n/p), (n - 1 - n/p)\}$. This means we want to have $\Delta(u - u_\infty - u_0) \in \text{Range } \Delta: W^{p,\delta}_{2,\delta} \to L^p_{\delta+2}$. By (7), this will be true if

$$\int_{\mathbb{R}^n} \Delta(u - u_\infty - u_0) \, dx = 0.$$  

By the Divergence Theorem and the fact that $u_0$ is harmonic for $|x| > 1$,

$$\int_{\mathbb{R}^n} \Delta u_0 \, dx = \int_{|x| \leq 1} \Delta u_0 \, dx = \int_{|x| = 1} \frac{\partial u_0}{\partial n} \, ds = (2 - n)\omega_n f_0.$$  

Therefore $f_0$ must satisfy

$$f_0 = -\frac{1}{(n - 2)\omega_n} \int_{\mathbb{R}^n} \Delta u \, dx = \frac{1}{(n - 2)\omega_n} \int_{\mathbb{R}^n} Ku^\sigma \, dx.$$  

This gives us

$$u - u_\infty - \frac{f_0}{|x|^{n-2}} = O(|x|^{\gamma}) \text{ as } |x| \to \infty \text{ for all } \gamma > \max\{2 - l, 1 - n\}.$$  

The process for obtaining more terms in the asymptotic expansion is essentially the same. We look for terms that are Kelvin transforms of homogeneous harmonic polynomials. Suppose $l > n + m$. Assume we have $u_\infty, u_0, u_1, \ldots, u_{m-1}$ so that $u_\infty$ is constant, $u_i \in C^\infty(\mathbb{R}^n)$,

$$u_i(x) = \frac{f_i}{|x|^{n-2+2i}} \text{ for } |x| \geq 1$$  

where $f_i$ is a homogeneous harmonic polynomial of degree $i$ and

$$u - u_\infty - u_0 - \cdots - u_{m-1} = O(|x|^{\gamma}) \text{ as } |x| \to \infty \text{ for all } \gamma > 2 - (n + m).$$  

Let $\mathcal{H}_m$ be the vector space of homogeneous harmonic polynomials of degree $m$. Choose a basis $\{H_1, \ldots, H_M\}$ for $\mathcal{H}_m$ such that $\int_{|x| = 1} H_i H_j \, ds = \delta_{ij}$. We want to find coefficients $a_1, \ldots, a_M$ so that the next term in the asymptotic expansion of $u$ is $u_m$ with

$$u_m = \frac{\sum a_i H_i}{|x|^{n-2+2m}} \text{ for } |x| \geq 1.$$  

To determine the $a_i$ we use the fact that we want

$$\Delta(u - u_\infty - u_0 - \cdots - u_m) \in \mathcal{R} = \text{Range } \Delta: W^{p,\delta}_{2,\delta} \to L^p_{\delta+2}$$  

for all $\delta$ satisfying

$$(n - 2 - \frac{n}{p}) + m < \delta < \min\{(l - 2 - \frac{n}{p}), (n - 2 - \frac{n}{p}) + m + 1\}.$$  

We have

$$\mathcal{R} = \left\{ g \in L^p_{\delta+2}: \int_{\mathbb{R}^n} g(x) H(x) \, dx = 0 \text{ for all } H \in \bigcup_{j=0}^m \mathcal{H}_j \right\}.$$  

so we need
\[ \int_{\mathbb{R}^n} \Delta(u - u_{\infty} - u_0 - \cdots - u_m)H(x) \, dx = 0 \quad \text{for all } H \in \bigcup_{j=0}^{m} \mathcal{H}_j. \]

Since for \( H \in \mathcal{H}_m \)
\[ \int_{\mathbb{R}^n} \Delta(c - u_0 - \cdots - u_{m-1})H(x) \, dx = 0, \]
it is sufficient to have
\[ \int_{\mathbb{R}^n} \Delta(u - u_m)H_j(x) \, dx = 0 \quad \text{for } j = 1, \ldots, M. \]

This gives us \( M \) conditions which determine \( a_1, \ldots, a_M \). A computation shows that
\[ \int_{\mathbb{R}^n} \Delta u H_i(x) \, dx = \int_{\mathbb{R}^n} \Delta u_m H_i(x) \, dx, \]
if
\[ a_i = \frac{1}{2 - n - 2m} \int_{\mathbb{R}^n} \Delta u H_i(x) \, dx = \frac{1}{2 - n - 2m} \int_{\mathbb{R}^n} -Ku^\sigma H_i(x) \, dx. \]

This completes the proof of Theorem 1.

4. PROOF OF THEOREM 2

Fix \( c > 0 \). Choose \( p > n \) and \( \sigma \) satisfying \( -n/p < \sigma < \min\{1, n\} - 2 - n/p \).

For each \( v \in W_{1,\delta}^p \), define
\[ f_v = \begin{cases} -K(v + c)^\sigma & \text{if } v + c \geq 0; \\ 0 & \text{if } v + c < 0. \end{cases} \]

Then \( f_v \in L_{\delta'+2}^p \) for \( \delta' < (l-2) - n/p \) due to the decay of \( K \). Therefore, for any \( \delta' \) satisfying \( \delta < \delta' < \min\{l, n\} - 2 - n/p \), we have

(i) \( f_v \in L_{\delta'+2}^p \) for all \( v \in W_{1,\delta}^p \)

(ii) \( \Delta: W_{2,\delta'}^p \to L_{\delta'+2}^p \) is an isomorphism by (5).

Therefore there is a unique \( w \in W_{2,\delta'}^p \) with \( \Delta w = f_v \). This yields a map
\[ T: W_{1,\delta}^p \to W_{2,\delta'}^p \]
given by \( Tv = w \) where \( \Delta w = f_v \). Since \( W_{2,\delta'}^p \subset W_{1,\delta}^p \) and the inclusion is compact, \( T \) is a compact mapping
\[ T: W_{1,\delta}^p \to W_{1,\delta}^p. \]

Suppose \( v = \alpha Tv \) for some \( \alpha \in [0, 1] \). This means that
\[ \Delta v = \begin{cases} -\alpha K(v + c)^\sigma & \text{if } v + c \geq 0; \\ 0 & \text{if } v + c < 0. \end{cases} \]
Since $K \leq 0$, $\Delta v \geq 0$ and since $v \in W^{p,\delta}_1$, $v \to 0$ as $|x| \to \infty$ by (4). By the maximum principle, $v \leq 0$ and $v + c \leq c$. Therefore

$$
\|v\|_{p,1,\delta} \leq \|v\|_{p,2,\delta} \leq A\|\Delta v\|_{L^{p+2}} \\
\leq A\|f_v\|_{L^{p+2}} \\
\leq Ac^\sigma\|K\|_{L^{p+2}}.
$$

The Leray-Schauder Fixed Point Theorem applies with $M = Ac^\sigma\|K\|_{L^{p+2}}$ and $T$ has a fixed point; that is, there is some $v \in W^{p,\delta}_1$ satisfying $\Delta v = f_v$. $K$ is locally Hölder continuous by hypothesis and $v$ is Hölder continuous because $v \in W^{p,\delta}_1$ and $p > n$. Therefore $f_v$ is Hölder continuous. Since $\Delta v = f_v$, elliptic regularity implies that $v \in C^2$. We shall show that $u = v + c$ is a solution to $\Delta u + Ku^\sigma = 0$. This is true if $v + c \geq 0$ for all $x$, for then $f_v = -K(v + c)^\sigma$ everywhere.

To show $v + c \geq 0$, assume $v + c < 0$ somewhere, and show this leads to a contradiction. Suppose $v$ has a minimum at $x_0$ with $v(x_0) + c = a < 0$. Since $v$ is continuous, there is some neighborhood $N$ of $x_0$ such that $v(x) + c < 0$ for all $x \in N$. Then for all $x \in N$, $\Delta v = 0$. Since $v$ attains its minimum at $x_0$, which is in the interior of $N$, $v$ must be constant on $N$. Therefore $v + c = a$ for all $x \in N$. Since $v$ is continuous, $v + c = a$ on $\partial N$. Continuing this, we would get that $v$ is constant on all of $\mathbb{R}^n$. Since $c > 0$ and $a < 0$, $v$ would have to be some negative constant on $\mathbb{R}^n$. However, we know that $v \to 0$ as $|x| \to \infty$ because $v \in W^{p,\delta}_1$. Therefore, $v + c \geq 0$ everywhere and $u = v + c$ is a solution to the equation. Since both $u$ and $K$ are bounded and $\sigma > 1$, we have $u$ satisfying $\Delta u - Au \leq 0$ for $A$ sufficiently large. By the Hopf maximum principle, $u$ cannot achieve a nonpositive minimum since we know $u$ is not identically zero. Therefore we have $u > 0$ on $\mathbb{R}^n$.

To complete the proof of Theorem 2, we must show uniqueness. Suppose $u_1$ and $u_2$ both satisfy $\Delta u + Ku^\sigma = 0$ and $u_1, u_2 \to c$ as $|x| \to \infty$. Then $u_1 - u_2 \to 0$ as $|x| \to \infty$. We shall show that $u - v$ cannot attain a positive maximum nor a negative minimum. Suppose $u_1 - u_2$ has a positive maximum at $x_0$. Then in a neighborhood of $x_0$, $u_1 - u_2 > 0$. Let $\mathcal{Z} = \{x: (u_1 - u_2)(x) > 0\}$. On the boundary of $\mathcal{Z}$, $u_1 - u_2$ must equal 0. In $\mathcal{Z}$, $\Delta(u_1 - u_2) = -K(u_1^\sigma - u_2^\sigma)$ is nonnegative, so $u_1 - u_2$ is subharmonic. Therefore, by the maximum principle, $u_1 - u_2$ must be constant in $\mathcal{Z}$. This would mean that $\mathcal{Z}$ must be all of $\mathbb{R}^n$. However, we know that $u_1 - u_2 \to 0$ as $|x| \to \infty$ and so cannot be a nonzero constant everywhere. Therefore $u_1 - u_2$ cannot have a positive maximum. Similarly, $u_1 - u_2$ cannot have a negative minimum. Since $u_1 - u_2 \to 0$ as $|x| \to \infty$, we must have $u_1 = u_2$. 


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