ON THE REGULARITY PROPERTIES FOR SOLUTIONS OF THE
CAUCHY PROBLEM FOR THE POROUS MEDIA EQUATION

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Abstract. We consider the Cauchy problem for the equation $\partial_t u = \Delta u^m$ in $\mathbb{R}^N \times (0, T)$. We assume that $1 < m < 3N/(3N - 2)$ and the initial data $u_0$ is in $C^1_0(\mathbb{R}^N)$ and $u_0 \geq 0$ in $\mathbb{R}^N$. Then we prove that the second derivatives of $u^m$ with respect to the space-variable are in $L^2(\mathbb{R}^N \times (0, T))$.

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We consider the Cauchy problem for the porous media equation

\begin{equation}
\begin{cases}
\partial_t u = \Delta u^m & \text{in } \mathbb{R}^N \times (0, T) \\
u(x, 0) = u_0(x) & \text{on } \mathbb{R}^N,
\end{cases}
\end{equation}

where $m > 1$ and $u_0(x) \geq 0$. The problem (1.1) has been studied by many authors. For a detailed account of (1.1) we refer to the work of Peletier [7].

We say that $u(x, t)$ is a solution of (1.1), if

\[
\int_0^T \int_{\mathbb{R}^N} [u(x, t)^2 + |\nabla_x u^m(x, t)|^2]dxdt < \infty
\]

and

\[
\int_0^T \int_{\mathbb{R}^N} (u \partial_t \phi - \nabla_x u^m \cdot \nabla_x \phi)dxdt + \int_{\mathbb{R}^N} u_0(x) \phi(x, 0)dx = 0
\]

for any continuously differentiable function $\phi(x, t)$ with compact support in $\mathbb{R}^N \times [0, T)$. The existence of such a solution is due to Sabinina [8] under some condition on $u_0$.

We are concerned with the regularity of $u$ in (1.1). The Hölder regularity of $u$ was shown by Caffarelli and Friedman [6]. For $N = 1$ the precise...
Hölder exponent with respect to the space-variable was obtained by Aronson [1]. Similar results for the time-variable were studied by di Benedetto [4], when \( N \geq 1 \). There arises a question whether the derivative \( \partial_t u \) is a function or not. Concerning this there are results such as Aronson and Bénilan [2], Bénilan [5], where the assumption on \( u_0 \) is very weak. For a function space \( A \) the assertion \( \partial_t u \in A \) is almost equivalent to \( \partial_{x_i} \partial_{x_j} u^m \in A, 1 \leq i, j \leq N \).

According to [5]

\[
\partial_t u \in L^p([\delta, T] \times B_R)
\]

for any \( 1 < p < 1 + 1/m, 0 < \delta < T \) and \( R > 0 \), where \( B_R = \{x \in \mathbb{R}^N; |x| < R\} \). Further if \( N = 1 \) and \( m > 2 \) particularly, \( \partial_t u \in L^\infty(\delta, T; L^p(B_R)) \) for any \( 1 < p < 1 + 1/(m - 2) \). In this connection we note also the results in [3], [9] and [10].

Our theorem is stated in the following section.

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Let us denote by \((\ , \ )\) and \( \| \| \) the inner product and the norm in \( L^2(\mathbb{R}^N) \), respectively. Let us write \( G^T = \mathbb{R}^N \times (0, T) \). Our aim is to prove

**Theorem.** Suppose \( 1 < m < 3N/(3N-2) \). Let \( u_0 \) be in \( C^1(\mathbb{R}^N) \), and let \( u_0 \geq 0 \) in \( \mathbb{R}^N \). Let \( u \) be a solution of (1.1). Then \( \partial_{x_i} \partial_{x_j} u^m \in L^2(G^T), 1 \leq i, j \leq N \).

More precisely, if \( u_0 \leq M \) in \( \mathbb{R}^N \), then for \( \alpha = 1 - 1/m \)

\[
\int_0^T \| \partial_{x_i} \partial_{x_j} u^m \|^2 dt \leq C[\| u_0^m \|^2 + \| u_0^{m(1-\alpha/2)} \|^2 + M^\alpha \| \nabla u_0^m \|^2 + (u_0^{-\alpha m}, |\nabla u_0^m|^2)],
\]

where \( C \) depends on \( m, N, T \) and not on \( u_0, M \).

Our method is to derive a uniform energy inequality for each solution of nondegenerate parabolic equations, which are the regular approximation of (1.1) appearing in [6].

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We prove our theorem.

For \( \eta > 0 \) let \( u_\eta(x, t) \) be the solution of

\[
\begin{cases}
\partial_t u_\eta = \Delta(u_\eta)^m & \text{in } \mathbb{R}^N \times (0, \infty) \\
u_\eta(x, 0) = u_0(x) + \eta & \text{on } \mathbb{R}^N.
\end{cases}
\]

It is known that \( \eta \leq u_\eta \leq M + \eta \) and \( u_\eta \) is classical (cf., e.g., [8]). If we set \( v_\eta = (u_\eta)^m \) and \( \psi_\eta = (u_0 + \eta)^m \), (3.1) becomes

\[
\begin{cases}
(v_\eta)^{-\alpha} \partial_t v_\eta = m\Delta v_\eta & \text{in } \mathbb{R}^N \times (0, \infty) \\
v_\eta(x, 0) = \psi_\eta(x) & \text{on } \mathbb{R}^N,
\end{cases}
\]
where $\alpha = 1 - 1/m$. For simplicity we denote $v^n(x,t)$ and $\psi^n(x)$ by $v(x,t)$ and $\psi(x)$, respectively. From the proof in Sabinina [8] we easily see that

$$
(3.2) \quad ||v^n(\cdot, t) - \eta^m|| \leq C(||\psi - \eta^m|| + (M + \eta)^{am/2}||\nabla \psi||), \quad 0 < t < T,
$$

where $C$ depends on $T$ and not on $\psi, \eta$ and $M$.

Let $\zeta \in C^2_0(R^N)$ and $\zeta \geq 0$ in $R^N$. From (3.1') we have

$$
\frac{1}{2 - \alpha}(\zeta, \partial_t v^{2-\alpha}) + m(\zeta, |\nabla_x v|^2) = \frac{m}{2}(\Delta \zeta, v^2).
$$

Hence

$$
\int_0^T (\zeta, |\nabla_x v|^2) dt = \frac{1}{m(2 - \alpha)}[(\zeta, \psi^{2-\alpha})
- (\zeta, v^{\cdot}, v^{2-\alpha})] + \frac{1}{2} \int_0^T (\Delta \zeta, v^2) dt.
$$

Combining this with (3.2) we obtain

$$
(3.3) \quad \int_0^T (\zeta, |\nabla_x v|^2) dt \leq C[(\zeta, \psi^{2-\alpha}) + ||\psi - \eta^m||^2 + (M + \eta)^{am}||\nabla \psi||^2 + \eta^{2m}].
$$

Let $w = -\zeta \partial_{x_j} v$ for $1 \leq j \leq N$. By integration by parts we have

$$
(3.4) \quad -(\Delta w, w) = (\zeta \nabla_x \partial_{x_j} v, \nabla_x \partial_{x_j} v)
+ (\partial_{x_j} \zeta \cdot \nabla_m v, \nabla_x \partial_{x_j} v) - (\nabla \zeta \cdot \nabla_x v, \partial_{x_j}^2 v),
$$

where we have assumed that $v(\cdot, t) \in C^3(R^N)$. But (3.4) is valid for any $v(\cdot, t) \in C^2(R^N)$ by taking an approximating sequence of $v$. Similarly we have

$$
(3.5) \quad (v^{\cdot - \alpha} \partial_t v, w) = -\alpha (v^{\cdot - \alpha - 1} \partial_{x_j} v \cdot \partial_t v, \zeta \partial_{x_j} v)
+ (v^{\cdot - \alpha} \partial_t \partial_{x_j} v, \zeta \partial_{x_j} v) + (v^{\cdot - \alpha} \partial_t v, \partial_{x_j} \zeta \cdot \partial_{x_j} v).
$$

It is easy to see that

$$
(v^{\cdot - \alpha - 1} \partial_{x_j} v \cdot \partial_t v, \zeta \partial_{x_j} v) = m(\zeta v^{-1} \Delta v, (\partial_{x_j} v)^2),
$$

$$
(v^{\cdot - \alpha} \partial_t \partial_{x_j} v, \zeta \partial_{x_j} v) = \frac{1}{2}(\zeta v^{-\alpha}, \partial_x (\partial_{x_j} v)^2)
$$

and

$$
(v^{\cdot - \alpha} \partial_t v, \partial_{x_j} \zeta \cdot \partial_{x_j} v) = m(\Delta v, \partial_{x_j} \zeta \cdot \partial_{x_j} v).
$$

Combining these equalities with (3.5), we have

$$
\int_0^T (v^{\cdot - \alpha} \partial_t v, w) dt \geq -\frac{\alpha m}{2} \int_0^T (\zeta v^{-1} \Delta v, (\partial_{x_j} v)^2) dt
- \frac{1}{2}(\zeta v^{-\alpha}, (\partial_{x_j} v)^2) + m \int_0^T (\Delta v, \partial_{x_j} \zeta \cdot \partial_{x_j} v) dt.
$$
From this and (3.4), (3.1'), it follows that

\[
(3.6) \int_0^T \| \zeta^{1/2} \nabla_x \partial_{x_j} v \|^2 dt \\
\leq \frac{1}{2m} (\zeta \psi^{-\alpha}, (\partial_{x_j} \psi)^2) + \frac{\alpha}{2} \int_0^T (\zeta v^{-1} \Delta v, (\partial_{x_j} v)^2) dt \\
- \int_0^T (\Delta v, \partial_{x_j} \zeta \cdot \partial_{x_j} v) dt - \int_0^T (\partial_{x_j} \zeta \cdot \nabla_x v, \nabla_x \partial_{x_j} v) dt \\
+ \int_0^T (\nabla \zeta \cdot \nabla_v \partial_{x_j} v, \partial_{x_j} v) dt.
\]

Now by integration by parts

\[
(\zeta^{-1} \partial_{x_j}^2 v, (\partial_{x_j} v)^2) = - (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3) \\
+ (\zeta v^{-2}, (\partial_{x_j} v)^4) - 2(\zeta v^{-1} \partial_{x_j}^2 v, (\partial_{x_j} v)^2).
\]

Thus we have

\[
(\zeta^{-1} \partial_{x_j}^2 v, (\partial_{x_j} v)^2) = \frac{1}{3} [ (\zeta v^{-2}, (\partial_{x_j} v)^4) - (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3) ],
\]

so that (3.6) becomes

\[
(3.7) \int_0^T \| \zeta^{1/2} \nabla_x \partial_{x_j} v \|^2 dt \\
\leq \frac{1}{2m} (\zeta \psi^{-\alpha}, (\partial_{x_j} \psi)^2) + \frac{\alpha}{6} \int_0^T (\zeta v^{-2}, (\partial_{x_j} v)^4) dt \\
- \frac{\alpha}{6} \int_0^T (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3) dt + \frac{\alpha}{2} \sum_{i \neq j} \int_0^T (\zeta v^{-1} \partial_{x_j}^2 v, (\partial_{x_j} v)^2) dt \\
- \int_0^T (\Delta v, \partial_{x_j} \zeta \cdot \partial_{x_j} v) dt - \int_0^T (\partial_{x_j} \zeta \cdot \nabla_x v, \nabla_x \partial_{x_j} v) dt \\
+ \int_0^T (\nabla \zeta \cdot \nabla_v \partial_{x_j} v, \partial_{x_j} v) dt.
\]

Here we estimate the quantity \((\zeta v^{-2}, (\partial_{x_j} v)^4)\). By integration by parts

\[
(\zeta v^{-2} \partial_{x_j} v, (\partial_{x_j} v)^3) = 2(\zeta v^{-2} \partial_{x_j} v, (\partial_{x_j} v)^3) \\
- 3(\zeta v^{-1} \partial_{x_j}^2 v, (\partial_{x_j} v)^2) - (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3).
\]

Hence

\[
(\zeta v^{-2}, (\partial_{x_j} v)^4) = 3(\zeta v^{-1} \partial_{x_j}^2 v, (\partial_{x_j} v)^2) + (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3)
\leq \frac{9}{2} (\zeta, (\partial_{x_j} v)^2) + \frac{1}{2} (\zeta v^{-2}, (\partial_{x_j} v)^4) + (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3),
\]

which implies that

\[
(\zeta v^{-2}, (\partial_{x_j} v)^4) \leq 9(\zeta, (\partial_{x_j} v)^2) + 2(\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3).
\]
Since \( 2(\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3) \leq \varepsilon (\zeta v^{-1}, (\partial_{x_j} v)^4) + \varepsilon^{-1}(\zeta^{-1}(\partial_{x_j} \zeta)^2, (\partial_{x_j} v)^2) \) for sufficiently small \( \varepsilon > 0 \), we obtain

\[
(3.8) \quad (\zeta v^{-2}, (\partial_{x_j} v)^4) \leq \frac{9}{1-\varepsilon} (\zeta, (\partial_{x_j} v)^2) + C(\varepsilon)(\zeta^{-1}(\partial_{x_j} \zeta)^2, (\partial_{x_j} v)^2).
\]

From this and Cauchy's inequality it follows that

\[
(3.9) \quad |(\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3)| \leq \frac{9\varepsilon}{2(1-\varepsilon)} (\zeta, (\partial_{x_j} v)^2) + C'(\varepsilon)(\zeta^{-1}(\partial_{x_j} \zeta)^2, (\partial_{x_j} v)^2).
\]

Now we define

\[
I_j = \frac{\alpha}{6} (\zeta v^{-2}, (\partial_{x_j} v)^4) - \frac{\alpha}{6} (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3)
\]

\[
+ \frac{\alpha}{2} \sum_{i \neq j} (\zeta v^{-1} \partial_{x_i} v, (\partial_{x_j} v)^2), \quad 1 \leq j \leq N.
\]

Using (3.9) and Cauchy's inequality we have for \( \delta > 0 \)

\[
I_j \leq \frac{\alpha}{4\delta} \left( \zeta, \left( \sum_{i \neq j} \partial_{x_i} v^2 \right)^2 \right) + \left( \frac{\alpha\delta}{4} + \frac{\alpha}{6} \right) (\zeta v^{-2}, (\partial_{x_j} v)^4)
\]

\[
+ \frac{3\alpha\varepsilon}{4(1-\varepsilon)} (\zeta, (\partial_{x_j} v)^2) + C(\zeta^{-1}(\partial_{x_j} \zeta)^2, (\partial_{x_j} v)^2).
\]

Easily, \( (\sum_{i \neq j} \partial_{x_i} v^2)^2 \leq (N-1) \sum_{i \neq j} (\partial_{x_i} v)^2 \). Thus by (3.8) it follows that

\[
(3.10) \quad \sum_j I_j \leq \left[ \frac{\alpha}{4\delta} (N-1)^2 + \frac{9}{1-\varepsilon} \left( \frac{\alpha\delta}{4} + \frac{\alpha}{6} \right) + \frac{3\alpha\varepsilon}{4(1-\varepsilon)} \right] (\zeta, \sum_j (\partial_{x_j} v)^2)
\]

\[
+ C \sum_j (\zeta^{-1}(\partial_{x_j} \zeta)^2, (\partial_{x_j} v)^2).
\]

Here we put \( \delta = (N-1)/3 \). Then

\[
\alpha\frac{\delta}{4} (N-1)^2 + 9 \left( \frac{\alpha\delta}{4} + \frac{\alpha}{6} \right) = 3\alpha N/2.
\]

From our assumption on \( m \) we see that \( 3\alpha N/2 < 1 \).

Next we estimate the remaining terms on the right-hand side of (3.7). For sufficiently small \( \varepsilon' > 0 \)

\[
\sum_j \left| (\Delta v, \partial_{x_j} \zeta \cdot \partial_{x_j} v) + (\partial_{x_j} \zeta \cdot \nabla_x v, \nabla_x \partial_{x_j} v) - (\nabla \zeta \cdot \nabla_x v, \partial_{x_j} v^2) \right|
\]

\[
\leq \varepsilon' \sum_j \| \varepsilon^{1/2} \nabla_x \partial_{x_j} v \|^2 + C(\varepsilon')(\zeta^{-1}|\nabla \zeta|^2, |\nabla_x v|^2).
\]

Therefore combining (3.7), (3.10) with the above, we conclude that

\[
\sum_j \int_0^T \| \zeta^{1/2} \nabla_x \partial_{x_j} v \|^2 dt \leq C[(\zeta \psi^{-\alpha}, |\nabla \psi|^2)
\]

\[
+ \int_0^T (\zeta^{-1}|\nabla \zeta|^2, |\nabla_x v|^2) dt]
\]
Let $\xi(x)$ be in $C^\infty_0(R^N)$, and let us put $\zeta = \xi^2$. Then from the above inequality and (3.3) it follows that for $1 \leq i, j \leq N$

(3.11)

\[
\int_0^T \|\xi^{1/2} \partial_{x_{i}} \partial_{x_{j}} v^n\|^2 dt \\
\leq C[(\xi(\psi^n)^{-\alpha}, |\nabla \psi^n|^2) \\
+ (|\nabla \xi|^2, (\psi^n)^{2-\alpha}) + ||\psi^n - \eta^m||^2 + (M + \eta^m)||\nabla \psi^n||^2 + \eta^{2m}] .
\]

As is well known, $v^n \downarrow v (\eta \downarrow 0)$ in $G$, where $v = u^m$ and $u$ is the solution of (1.1). For each positive integer $n$ we put $\xi(x) = \xi_n(x)$, where

\[
\xi_n(x) = \begin{cases} 
1 & (|x| \leq n) \\
0 & (|x| > 2n)
\end{cases}
\]

and they are uniformly bounded in $R^N$ up to the third derivatives. Then the constant $C$ on the right-hand side of (3.11) are independent of $n$. Letting $\eta \to 0$ in (3.11), we see that $\partial_{x_{i}} \partial_{x_{j}} v^n \to \partial_{x_{i}} \partial_{x_{j}} v$ weakly in $L^2_{\text{loc}}(G)$ and

$\int_0^T \|\xi^{1/2} \partial_{x_{i}} \partial_{x_{j}} v\|^2 dt \leq$ the right-hand side of (2.1). This completes the proof.

References


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