

ON THE REGULARITY PROPERTIES FOR SOLUTIONS OF THE CAUCHY PROBLEM FOR THE POROUS MEDIA EQUATION

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(Communicated by Walter D. Littman)

ABSTRACT. We consider the Cauchy problem for the equation $\partial_t u = \Delta u^m$ in $R^N \times (0, T)$. We assume that $1 < m < 3N/(3N - 2)$ and the initial data u_0 is in $C_0^1(R^N)$ and $u_0 \geq 0$ in R^N . Then we prove that the second derivatives of u^m with respect to the space-variable are in $L^2(R^N \times (0, T))$.

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We consider the Cauchy problem for the porous media equation

$$(1.1) \quad \begin{cases} \partial_t u = \Delta u^m & \text{in } R^N \times (0, T) \\ u(x, 0) = u_0(x) & \text{on } R^N, \end{cases}$$

where $m > 1$ and $u_0(x) \geq 0$. The problem (1.1) has been studied by many authors. For a detailed account of (1.1) we refer to the work of Peletier [7].

We say that $u(x, t)$ is a solution of (1.1), if

$$\int_0^T \int_{R^N} [u(x, t)^2 + |\nabla_x u^m(x, t)|^2] dx dt < \infty$$

and

$$\begin{aligned} \int_0^T \int_{R^N} (u \partial_t \phi - \nabla_x u^m \cdot \nabla_x \phi) dx dt \\ + \int_{R^N} u_0(x) \phi(x, 0) dx = 0 \end{aligned}$$

for any continuously differentiable function $\phi(x, t)$ with compact support in $R^N \times [0, T)$. The existence of such a solution is due to Sabinina [8] under some condition on u_0 .

We are concerned with the regularity of u in (1.1). The Hölder regularity of u was shown by Caffarelli and Friedman [6]. For $N = 1$ the precise

Received by the editors September 21, 1987 and, in revised form, June 30, 1988.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 35K55, 35K65, 35K15.

Key words and phrases. Porous media, initial data.

This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education.

Hölder exponent with respect to the space-variable was obtained by Aronson [1]. Similar results for the time-variable were studied by di Benedetto [4], when $N \geq 1$. There arises a question whether the derivative $\partial_t u$ is a function or not. Concerning this there are results such as Aronson and Bénilan [2], Bénilan [5], where the assumption on u_0 is very weak. For a function space A the assertion “ $\partial_t u \in A$ ” is almost equivalent to “ $\partial_{x_i} \partial_{x_j} u^m \in A, 1 \leq i, j \leq N$ ”.

According to [5]

$$\partial_t u \in L^p([\delta, T] \times B_R)$$

for any $1 < p < 1 + 1/m, 0 < \delta < T$ and $R > 0$, where $B_R = \{x \in \mathbb{R}^N; |x| < R\}$. Further if $N = 1$ and $m > 2$ particularly, $\partial_t u \in L^\infty(\delta, T; L^p(B_R))$ for any $1 < p < 1 + 1/(m - 2)$. In this connection we note also the results in [3], [9] and [10].

Our theorem is stated in the following section.

2

Let us denote by (\cdot, \cdot) and $\|\cdot\|$ the inner product and the norm in $L^2(\mathbb{R}^N)$, respectively. Let us write $G^T = \mathbb{R}^N \times (0, T)$. Our aim is to prove

Theorem. *Suppose $1 < m < 3N/(3N - 2)$. Let u_0 be in $C_0^1(\mathbb{R}^N)$, and let $u_0 \geq 0$ in \mathbb{R}^N . Let u be a solution of (1.1). Then $\partial_{x_i} \partial_{x_j} u^m \in L^2(G^T), 1 \leq i, j \leq N$. More precisely, if $u_0 \leq M$ in \mathbb{R}^N , then for $\alpha = 1 - 1/m$*

$$(2.1) \quad \int_0^T \|\partial_{x_i} \partial_{x_j} u^m\|^2 dt \leq C[\|u_0^m\|^2 + \|u_0^{m(1-\alpha/2)}\|^2 + M^{\alpha m} \|\nabla u_0^m\|^2 + (u_0^{-\alpha m}, |\nabla u_0^m|^2)],$$

where C depends on m, N, T and not on u_0, M .

Our method is to derive a uniform energy inequality for each solution of nondegenerate parabolic equations, which are the regular approximation of (1.1) appearing in [6].

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We prove our theorem.

For $\eta > 0$ let $u^\eta(x, t)$ be the solution of

$$(3.1) \quad \begin{cases} \partial_t u^\eta = \Delta(u^\eta)^m & \text{in } \mathbb{R}^N \times (0, \infty) \\ u^\eta(x, 0) = u_0(x) + \eta & \text{on } \mathbb{R}^N. \end{cases}$$

It is known that $\eta \leq u^\eta \leq M + \eta$ and u^η is classical (cf., e.g., [8]). If we set $v^\eta = (u^\eta)^m$ and $\psi^\eta = (u_0 + \eta)^m$, (3.1) becomes

$$(3.1') \quad \begin{cases} (v^\eta)^{-\alpha} \partial_t v^\eta = m \Delta v^\eta & \text{in } \mathbb{R}^N \times (0, \infty) \\ v^\eta(x, 0) = \psi^\eta(x) & \text{on } \mathbb{R}^N, \end{cases}$$

where $\alpha = 1 - 1/m$. For simplicity we denote $v^n(x, t)$ and $\psi^n(x)$ by $v(x, t)$ and $\psi(x)$, respectively. From the proof in Sabinina [8] we easily see that

$$(3.2) \quad \|v^\eta(\cdot, t) - \eta^m\| \leq C(\|\psi - \eta^m\| + (M + \eta)^{\alpha m/2} \|\nabla \psi\|), \quad 0 < t < T,$$

where C depends on T and not on ψ, η and M .

Let $\zeta \in C_0^2(R^N)$ and $\zeta \geq 0$ in R^N . From (3.1') we have

$$\frac{1}{2 - \alpha} (\zeta, \partial_t v^{2-\alpha}) + m(\zeta, |\nabla_x v|^2) = \frac{m}{2} (\Delta \zeta, v^2).$$

Hence

$$\int_0^T (\zeta, |\nabla_x v|^2) dt = \frac{1}{m(2 - \alpha)} [(\zeta, \psi^{2-\alpha}) - (\zeta, v(\cdot, T)^{2-\alpha})] + \frac{1}{2} \int_0^T (\Delta \zeta, v^2) dt.$$

Combining this with (3.2) we obtain

$$(3.3) \quad \int_0^T (\zeta, |\nabla_x v|^2) dt \leq C[(\zeta, \psi^{2-\alpha}) + \|\psi - \eta^m\|^2 + (M + \eta)^{\alpha m} \|\nabla \psi\|^2 + \eta^{2m}].$$

Let $w = -\zeta \partial_{x_j}^2 v$ for $1 \leq j \leq N$. By integration by parts we have

$$(3.4) \quad -(\Delta v, w) = (\zeta \nabla_x \partial_{x_j} v, \nabla_x \partial_{x_j} v) + (\partial_{x_j} \zeta \cdot \nabla_m v, \nabla_x \partial_{x_j} v) - (\nabla \zeta \cdot \nabla_x v, \partial_{x_j}^2 v),$$

where we have assumed that $v(\cdot, t) \in C^3(R^N)$. But (3.4) is valid for any $v(\cdot, t) \in C^2(R^N)$ by taking an approximating sequence of v . Similarly we have

$$(3.5) \quad (v^{-\alpha} \partial_t v, w) = -\alpha (v^{-\alpha-1} \partial_{x_j} v \cdot \partial_t v, \zeta \partial_{x_j} v) + (v^{-\alpha} \partial_t \partial_{x_j} v, \zeta \partial_{x_j} v) + (v^{-\alpha} \partial_t v, \partial_{x_j} \zeta \cdot \partial_{x_j} v).$$

It is easy to see that

$$(v^{-\alpha-1} \partial_{x_j} v \cdot \partial_t v, \zeta \partial_{x_j} v) = m(\zeta v^{-1} \Delta v, (\partial_{x_j} v)^2),$$

$$(v^{-\alpha} \partial_t \partial_{x_j} v, \zeta \partial_{x_j} v) = \frac{1}{2} (\zeta v^{-\alpha}, \partial_t (\partial_{x_j} v)^2)$$

and

$$(v^{-\alpha} \partial_t v, \partial_{x_j} \zeta \cdot \partial_{x_j} v) = m(\Delta v, \partial_{x_j} \zeta \cdot \partial_{x_j} v).$$

Combining these equalities with (3.5), we have

$$\int_0^T (v^{-\alpha} \partial_t v, w) dt \geq -\frac{\alpha m}{2} \int_0^T (\zeta v^{-1} \Delta v, (\partial_{x_j} v)^2) dt - \frac{1}{2} (\zeta \psi^{-\alpha}, (\partial_{x_j} \psi)^2) + m \int_0^T (\Delta v, \partial_{x_j} \zeta \cdot \partial_{x_j} v) dt.$$

From this and (3.4), (3.1'), it follows that

$$\begin{aligned}
 (3.6) \quad & \int_0^T \|\zeta^{1/2} \nabla_x \partial_{x_j} v\|^2 dt \\
 & \leq \frac{1}{2m} (\zeta \psi^{-\alpha}, (\partial_{x_j} \psi)^2) + \frac{\alpha}{2} \int_0^T (\zeta v^{-1} \Delta v, (\partial_{x_j} v)^2) dt \\
 & \quad - \int_0^T (\Delta v, \partial_{x_j} \zeta \cdot \partial_{x_j} v) dt - \int_0^T (\partial_{x_j} \zeta \cdot \nabla_x v, \nabla_x \partial_{x_j} v) dt \\
 & \quad + \int_0^T (\nabla \zeta \cdot \nabla_x v, \partial_{x_j}^2 v) dt.
 \end{aligned}$$

Now by integration by parts

$$\begin{aligned}
 (\zeta v^{-1} \partial_{x_j}^2 v, (\partial_{x_j} v)^2) &= -(\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3) \\
 & \quad + (\zeta v^{-2}, (\partial_{x_j} v)^4) - 2(\zeta v^{-1} \partial_{x_j}^2 v, (\partial_{x_j} v)^2).
 \end{aligned}$$

Thus we have

$$(\zeta v^{-1} \partial_{x_j}^2 v, (\partial_{x_j} v)^2) = \frac{1}{3} [(\zeta v^{-2}, (\partial_{x_j} v)^4) - (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3)],$$

so that (3.6) becomes

$$\begin{aligned}
 (3.7) \quad & \int_0^T \|\zeta^{1/2} \nabla_x \partial_{x_j} v\|^2 dt \\
 & \leq \frac{1}{2m} (\zeta \psi^{-\alpha}, (\partial_{x_j} \psi)^2) + \frac{\alpha}{6} \int_0^T (\zeta v^{-2}, (\partial_{x_j} v)^4) dt \\
 & \quad - \frac{\alpha}{6} \int_0^T (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3) dt + \frac{\alpha}{2} \sum_{i \neq j} \int_0^T (\zeta v^{-1} \partial_{x_i}^2 v, (\partial_{x_j} v)^2) dt \\
 & \quad - \int_0^T (\Delta v, \partial_{x_j} \zeta \cdot \partial_{x_j} v) dt - \int_0^T (\partial_{x_j} \zeta \cdot \nabla_x v, \nabla_x \partial_{x_j} v) dt \\
 & \quad + \int_0^T (\nabla \zeta \cdot \nabla_x v, \partial_{x_j}^2 v) dt.
 \end{aligned}$$

Here we estimate the quantity $(\zeta v^{-2}, (\partial_{x_j} v)^4)$. By integration by parts

$$\begin{aligned}
 (\zeta v^{-2} \partial_{x_j} v, (\partial_{x_j} v)^3) &= 2(\zeta v^{-2} \partial_{x_j} v, (\partial_{x_j} v)^3) \\
 & \quad - 3(\zeta v^{-1} \partial_{x_j}^2 v, (\partial_{x_j} v)^2) - (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3).
 \end{aligned}$$

Hence

$$\begin{aligned}
 (\zeta v^{-2}, (\partial_{x_j} v)^4) &= 3(\zeta v^{-1} \partial_{x_j}^2 v, (\partial_{x_j} v)^2) + (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3) \\
 &\leq \frac{9}{2} (\zeta, (\partial_{x_j}^2 v)^2) + \frac{1}{2} (\zeta v^{-2}, (\partial_{x_j} v)^4) + (\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3),
 \end{aligned}$$

which implies that

$$(\zeta v^{-2}, (\partial_{x_j} v)^4) \leq 9(\zeta, (\partial_{x_j}^2 v)^2) + 2(\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3).$$

Since $2(\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3) \leq \varepsilon(\zeta v^{-1}, (\partial_{x_j} v)^4) + \varepsilon^{-1}(\zeta^{-1}(\partial_{x_j} \zeta)^2, (\partial_{x_j} v)^2)$ for sufficiently small $\varepsilon > 0$, we obtain

$$(3.8) \quad (\zeta v^{-2}, (\partial_{x_j} v)^4) \leq \frac{9}{1-\varepsilon}(\zeta, (\partial_{x_j}^2 v)^2) + C(\varepsilon)(\zeta^{-1}(\partial_{x_j} \zeta)^2, (\partial_{x_j} v)^2).$$

From this and Cauchy’s inequality it follows that

$$(3.9) \quad |(\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3)| \leq \frac{9\varepsilon}{2(1-\varepsilon)}(\zeta, (\partial_{x_j}^2 v)^2) + C'(\varepsilon)(\zeta^{-1}(\partial_{x_j} \zeta)^2, (\partial_{x_j} v)^2).$$

Now we define

$$I_j = \frac{\alpha}{6}(\zeta v^{-2}, (\partial_{x_j} v)^4) - \frac{\alpha}{6}(\partial_{x_j} \zeta \cdot v^{-1}, (\partial_{x_j} v)^3) + \frac{\alpha}{2} \sum_{i \neq j} (\zeta v^{-1} \partial_{x_i}^2 v, (\partial_{x_j} v)^2), \quad 1 \leq j \leq N.$$

Using (3.9) and Cauchy’s inequality we have for $\delta > 0$

$$I_j \leq \frac{\alpha}{4\delta} \left(\zeta, \left(\sum_{i \neq j} \partial_{x_i}^2 v \right)^2 \right) + \left(\frac{\alpha\delta}{4} + \frac{\alpha}{6} \right) (\zeta v^{-2}, (\partial_{x_j} v)^4) + \frac{3\alpha\varepsilon}{4(1-\varepsilon)} (\zeta, (\partial_{x_j}^2 v)^2) + C(\zeta^{-1}(\partial_{x_j} \zeta)^2, (\partial_{x_j} v)^2).$$

Easily, $(\sum_{i \neq j} \partial_{x_i}^2 v)^2 \leq (N-1) \sum_{i \neq j} (\partial_{x_i}^2 v)^2$. Thus by (3.8) it follows that

$$(3.10) \quad \sum_j I_j \leq \left[\frac{\alpha}{4\delta}(N-1)^2 + \frac{9}{1-\varepsilon} \left(\frac{\alpha\delta}{4} + \frac{\alpha}{6} \right) + \frac{3\alpha\varepsilon}{4(1-\varepsilon)} \right] \left(\zeta, \sum_j (\partial_{x_j}^2 v)^2 \right) + C \sum_j (\zeta^{-1}(\partial_{x_j} \zeta)^2, (\partial_{x_j} v)^2).$$

Here we put $\delta = (N-1)/3$. Then

$$\frac{\alpha}{4\delta}(N-1)^2 + 9 \left(\frac{\alpha\delta}{4} + \frac{\alpha}{6} \right) = 3\alpha N/2.$$

From our assumption on m we see that $3\alpha N/2 < 1$.

Next we estimate the remaining terms on the right-hand side of (3.7). For sufficiently small $\varepsilon' > 0$

$$\sum_j |(\Delta v, \partial_{x_j} \zeta \cdot \partial_{x_j} v) + (\partial_{x_j} \zeta \cdot \nabla_x v, \nabla_x \partial_{x_j} v) - (\nabla \zeta \cdot \nabla_x v, \partial_{x_j}^2 v)| \leq \varepsilon' \sum_j \|\zeta^{1/2} \nabla_x \partial_{x_j} v\|^2 + C(\varepsilon')(\zeta^{-1}|\nabla \zeta|^2, |\nabla_x v|^2).$$

Therefore combining (3.7), (3.10) with the above, we conclude that

$$\sum_j \int_0^T \|\zeta^{1/2} \nabla_x \partial_{x_j} v\|^2 dt \leq C[(\zeta \psi^{-\alpha}, |\nabla \psi|^2) + \int_0^T (\zeta^{-1}|\nabla \zeta|^2, |\nabla_x v|^2) dt].$$

Let $\xi(x)$ be in $C_0^\infty(R^N)$, and let us put $\zeta = \xi^2$. Then from the above inequality and (3.3) it follows that for $1 \leq i, j \leq N$

(3.11)

$$\begin{aligned} & \int_0^T \|\zeta^{1/2} \partial_{x_i} \partial_{x_j} v^\eta\|^2 dt \\ & \leq C[(\zeta(\psi^\eta)^{-\alpha}, |\nabla \psi^\eta|^2) \\ & \quad + (|\nabla \xi|^2, (\psi^\eta)^{2-\alpha}) + \|\psi^\eta - \eta^m\|^2 + (M + \eta)^{\alpha m} \|\nabla \psi^\eta\|^2 + \eta^{2m}]. \end{aligned}$$

As is well known, $v^\eta \downarrow v$ ($\eta \downarrow 0$) in G , where $v = u^m$ and u is the solution of (1.1). For each positive integer n we put $\xi(x) = \xi_n(x)$, where

$$\xi_n(x) = \begin{cases} 1 & (|x| \leq n) \\ 0 & (|x| > 2n) \end{cases}$$

and they are uniformly bounded in R^N up to the third derivatives. Then the constant C on the right-hand side of (3.11) are independent of n . Letting $\eta \rightarrow 0$ in (3.11), we see that $\partial_{x_i} \partial_{x_j} v^\eta \rightarrow \partial_{x_i} \partial_{x_j} v$ weakly in $L_{\text{loc}}^2(G)$ and $\int_0^T \|\zeta^{1/2} \partial_{x_i} \partial_{x_j} v\|^2 dt \leq$ the right-hand side of (2.1). This completes the proof.

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