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ON THE THEOREMS OF ŠARKOVSKIĬ AND ŠTEFAN ON CYCLES

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ABSTRACT. New proofs of the well-known theorems of Šarkovskiĭ and Štefan on cycles of a continuous real mapping are given.

Let us fix a real interval I and a continuous function $f: I \to I$. For any $n \in \mathbf{N}$ we denote by f^n the n-th iterate of f. A point $x \in I$ is said to be a periodic point of f if x is a fixed point of f^n for some $n \in \mathbf{N}$. If x is periodic, the smallest $n \in \mathbf{N}$ with $f^n(x) = x$ is called the period of x. Throughout this paper, $\mathrm{Per}(f, n)$ will stand for the set of all periodic points of f of the period n.

Let us consider the following ordering of the set $\mathbf{N}$:

$$3 \dashv 5 \dashv 7 \dashv \cdots \dashv 2\cdot 3 \dashv 2\cdot 5 \dashv 2\cdot 7 \dashv \cdots \dashv 2^n\cdot 3 \dashv 2^n\cdot 5 \dashv \cdots \dashv 2^n \dashv \cdots \dashv 2^2 \dashv 2 \dashv 1.$$

The aim of this paper is to give new proofs of the following two theorems.

Theorem A (A. N. Šarkovskiĭ). *Let* $n \in \mathbf{N}$. *If* $\mathrm{Per}(f, n) \neq \varnothing$ *then* $\mathrm{Per}(f, m) \neq \varnothing$ *for any* $m \in \mathbf{N}$ *such that* $n \dashv m$.

Theorem B (P. Štefan). *Let* $n \geq 3$ *be odd. Assume that* $\mathrm{Per}(f, m) = \varnothing$ *for any* $m \dashv n$. *If* $x_0 \in \mathrm{Per}(f, n)$ *then there exists a point* $x \in \{x_0, \dots, f^{n-1}(x_0)\}$ *such that*

$$x < f^{n-2}(x) < \cdots < f(x) < f^2(x) < \cdots < f^{n-1}(x)$$

or

$$x > f^{n-2}(x) > \cdots > f(x) > f^2(x) > \cdots > f^{n-1}(x).$$

Theorem A is a well-known result proved originally by A. N. Šarkovskiĭ [5]. Another proof of Theorem A was given by P. Štefan in [6] who filled some gaps in Šarkovskiĭ's argument. Making use of some parts of his proof (cf. [6, Lemmas (20) and (21)]) Štefan proved Theorem B, which shows how f acts on its minimal orbits of odd period. Chung-Wu Ho and Ch. Morris [4] also proved Theorems A and B by following some ideas of P. D. Straffin Jr. concerning directed graphs (cf. [7]). Another proof of Theorem A using the work of Straffin and digraphs was given by U. Burkart [3]. A sketch of a proof

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of Theorem of Šarkovskiĭ may be found also in the paper [2]. Some comments concerning the history of the Šarkovskiĭ's result as well as the Burkart's proof have been presented with details by Gy. Targonski in his monograph [8, Ch. 8, §2].

Up to now the greatest difficulty in proving Theorem A has been to prove it in the particular case where n is odd (Lemma 8 below). Usually it takes the largest parts of the proofs (cf. for example [5, Theorem 4 and Lemmas 5 and 6] or [6, (15), (16), (18) and Section D]). We proceed in a different, more natural way starting with a direct proof of Theorem B. Then we can simply prove Lemma 8 making use of the following three facts: Theorem B, Lemma 6 (which is actually a result of L. Block [1]), and a simple Lemma 7. Consequently we obtain another proof of Theorem B and a new, rather short and clear proof of the Theorem of Šarkovskiĭ. Lemmas 1, 2, 3, 4, 9, and 10 are standard and may be found for example in [6]. Nevertheless we prove them (Lemma 3 in a shorter way) for the convenience of the reader. Also Lemma 6 is presented with a short argument.

Lemma 1. *Assume that* $m, n \in \mathbf{N}$.

(i) $x \in \operatorname{Per}(f^n, 1)$ *iff* $x \in \operatorname{Per} f$ *and the order of* x *divides* n.

(ii) $\operatorname{Per}(f, n) \subset \operatorname{Per}(f^m, n/k)$ *where* k *is the highest common divisor of* n *and* m.

(iii) *If* p *is prime then* $\operatorname{Per}(f, p^{n+m}) = \operatorname{Per}(f^{p^n}, p^m)$.

Proof of Theorem B. If $n = 3$, the statement is evident. Thus we can assume that $n \geq 5$. Let $C = \{x_0, \dots, f^{n-1}(x_0)\}$. Since n is odd, there exists a point $y \in C$ such that

$$y < f(y) < f^2(y) \quad \text{or} \quad y > f(y) > f^2(y).$$

Assume for example the first of the above conditions. Put

$$x = \min\{y \in C : y < f(y) < f^2(y)\} \quad \text{and} \quad y = f^{n-1}(x).$$

Clearly $x < f(x) < f^2(x)$ and $x < y$. We show that $f^2(x) < y$. Suppose on the contrary that $y \in (x, f(x)) \cup (f(x), f^2(x))$. Consider the case $y \in (x, f(x))$. Then

$$f(y) = x < y < f(x),$$

whence $\operatorname{Per}(f, 1) \cap (x, y) \neq \emptyset$. Let a be its maximal point. Since

$$f^2(a) = a < y < f(x) = f^2(y)$$

there exists a $z_1 \in (a, y)$ such that $f^2(z_1) = y$. Thus

$$f^3(z_1) = x < z_1 \quad \text{and} \quad y < f^2(x) = f^3(y)$$

which (cf. the assumption of the theorem, Lemma 1(i), and the definition of a) is impossible. Now suppose that $y \in (f(x), f^2(x))$. Put $a = \sup \operatorname{Per}(f, 1) \cap (f(x), y)$. Since

$$f(a) = a < y < f(f(x))$$

there is a $z_1 \in (f(x), a)$ such that $f(z_1) = y$. Due to the inequalities

$$f(y) = x < z_1 < a = f(a)$$

we can find a $z_2 \in (a, y)$ for which $f(z_2) = z_1$. Therefore we have

$$f^3(z_2) = x < z_2 \quad \text{and} \quad y < f^2(x) = f^3(y)$$

which again leads to a contradiction. We now have that $f^2(x) < y$.

Let us observe that, by the inequalities

$$f(x) < y = f^{n-2}(f(x)), \qquad f^{n-2}(f^2(x)) = x < f^2(x),$$

Lemma 1(i) and the assumption of the theorem, $\mathrm{Per}(f, 1) \cap (f(x), f^2(x)) \neq \varnothing$. Put

$$a_1 = \inf \mathrm{Per}(f, 1) \cap (f(x), f^2(x))$$

and

$$a_2 = \sup \mathrm{Per}(f, 1) \cap (f(x), f^2(x)).$$

Now, we will show, that the following condition is fulfilled for any $k \in \{1, \dots, (n-1)/2\}$

(1) if $f^{2k-1}(x) < \cdots < f(x) < f^2(x) < \cdots < f^{2k}(x)$, then $((f^{2k-1}(x), a_1) \cup (a_2, f^{2k}(x))) \cap \mathrm{Per}(f, 1) = \varnothing$.

For fix a $k \in \{1, \dots, (n-1)/2\}$ and suppose on the contrary that (1) does not hold. Clearly we may assume that $k \geq 2$. At first let a be a point of $\mathrm{Per}(f, 1) \cap (f^{2k-1}(x), f(x))$. From the inequality

$$f^{2k-2}(f(x)) = f^{2k-1}(x) < a < a_1 = f^{2k-2}(a_1)$$

we deduce, that $f^{2k-2}(z_1) = a$ for a $z_1 \in (f(x), a_1)$. Hence

$$f(x) < f^{2k}(x) = f^{2k-1}(f(x))$$

and

$$f^{2k-1}(z_1) = f(f^{2k-2}(z_1)) = a < z_1,$$

which contradicts our assumptions. Now suppose that $\mathrm{Per}(f, 1) \cap (f^{2i}(x), f^{2i+2}(x)) \neq \varnothing$ for an $i \in \{1, \dots, k-1\}$. Fix an element a of this set. Since

$$f^{2i}(a_2) = a_2 < a < f^{2i+2}(x) = f^{2i}(f^2(x)),$$

there exists a $z_1 \in (a_2, f^2(x))$ such that $f^{2i}(z_1) = a$ which, in view of the inequalities

$$f^{n-2}(f^2(x)) = x < f^2(x) \quad \text{and} \quad z_1 < a = f^{n-2i-2}(a) = f^{n-2}(z_1),$$

contradicts our assumptions and finishes the proof of (1).

To complete the proof of Theorem B it is enough to show inductively that, for any $k \in \{1, \dots, (n-1)/2\}$

(2) $$x < f^{2k-1}(x) < \cdots < f(x) < f^2(x) < \cdots < f^{2k}(x) \leq f^{n-1}(x) = y.$$

If $k = 1$ then (2) follows from the definition of x. Fix a $k \in \{1, \dots, (n-3)/2\}$ and assume (2). At first we shall show that $f^{2k+1}(x) \in (x, f^{2k-1}(x))$. Suppose that $f^{2k+1}(x) > f^{2k}(x)$. Then for an $a \in \operatorname{Per}(f, 1) \cap (f^{2k}(x), y)$,

$$f^{n-3}(a_2) = a_2 < a < y = f^{n-3}(f^2(x))$$

whence there exists a $z_1 \in (a_2, f^2(x))$ such that $f^{n-3}(z_1) = a$. Therefore

$$f^{n-2}(f^2(x)) = x < f^2(x) \quad \text{and} \quad z_1 < a = f(a) = f(f^{n-3}(z_1)) = f^{n-2}(z_1)$$

which leads to a contradiction and proves that $f^{2k+1}(x) < f^{2k}(x)$. Now suppose that $f^{2k+1}(x) \in (a_1, f^{2k}(x))$. From the inequalities

$$f^{2k-1}(a_1) = a_1 < f^{2k+1}(x) < f^{2k}(x) = f^{2k-1}(f(x))$$

we deduce that there exists a $z_1 \in (f(x), a_1)$ such that $f^{2k-1}(z_1) = f^{2k+1}(x)$. Since

$$f^{n-2}(z_1) = f^{n-2k-1}(f^{2k-1}(z_1)) = f^{n-2k-1}(f^{2k+1}(x)) = x < z_1$$

and

$$f(x) < y = f^{n-2}(f(x)),$$

there is a fixed point of f^{n-2} in $(f(x), z_1)$ which is impossible. If $f^{2k+1}(x) \in (f^{2k-1}(x), a_1)$ then by the inequalities

$$f^{n-2}(f^{2k+1}(x)) = f^{2k-1}(x) < f^{2k+1}(x)$$

and

$$f(x) < y = f^{n-2}(f(x)),$$

the interval with the endpoints $f(x)$ and $f^{2k+1}(x)$ would contain a fixed point of f^{n-2} and this contradicts condition (1). At last suppose that $f^{2k+1}(x) < x$. Putting $a = \min(\{f(x)\} \cup (\operatorname{Per}(f, 1) \cap (x, f(x))))$ we have

$$f^2(a) < a_1 < f^2(x),$$

whence $f^2(z_1) = a_1$ for a $z_1 \in (x, a)$. Thus

$$f^{2k+1}(x) < x \quad \text{and} \quad f^{2k+1}(z_1) = a_1 > z_1,$$

which is impossible. Consequently, $f^{2k+1}(x) \in (x, f^{2k-1}(x))$.

Now we shall show that $f^{2k+2}(x) \in (f^{2k}(x), y]$. Clearly we may assume that $k \le (n-5)/2$. At first suppose that $y < f^{2k+2}(x)$. In view of the inequalities

$$f^{2k+1}(a_1) = a_1 < y < f^{2k+2}(x) = f^{2k+1}(f(x))$$

there exists a $z_1 \in (f(x), a_1)$ such that $f^{2k+1}(z_1) = y$. Since

$$f^2(f(x)) < f(x) < z_1 < a_1 = f^2(a_1),$$

$z_1 = f^2(z_2)$ for a $z_2 \in (f(x), a_1)$. Thus

$$f^{2k+3}(z_1) = f^2(y) = f(x) < z_1$$

and

$$z_2 < y = f^{2k+1}(z_1) = f^{2k+3}(z_2),$$

which is false and proves that $f^{2k+2}(x) \le y$. Suppose that $f^{2k+2}(x) < f(x)$. Since

$$f^2(f(x)) < f(x) < a_1 = f^2(a_1),$$

there exists a $z_1 \in (f(x), a_1)$ such that $f^2(z_1) = f(x)$. Then we have

$$f^{2k+1}(f(x)) = f^{2k+2}(x) < f(x)$$

and

$$z_1 < f^{2k}(x) = f^{2k-1}(f(x)) = f^{2k+1}(z_1),$$

which is impossible. Hence $f^{2k+2}(x) > f(x)$. Now we shall prove that

(3) $$(f^{2k+1}(x), f^{2k-1}(x)) \cap \mathrm{Per}(f, 1) = \varnothing.$$

Indeed, suppose that this is not the case and put $a = \min \mathrm{Per}(f, 1) \cap (f^{2k+1}(x), f^{2k-1}(x))$. Since

$$f(a) = a < f^{2k-1}(x) \le f(x) < a_2 < f^{2k}(x) = f(f^{2k-1}(x)),$$

there exist a $z_1 \in (a, f^{2k-1}(x))$ and $z_2, z_3 \in (f^{2k+1}(x), a)$ such that $f(z_1) = a_2$, $f(z_2) = z_1$, $f(z_3) = f^{2k-1}(x)$. Then

$$f^3(z_2) = f^2(z_1) = a_2 > z_2, \qquad f^3(z_3) = f^2(f^{2k-1}(x)) = f^{2k+1}(x) < z_3,$$

contrary to the definition of a. This completes the proof of (3). Suppose that $f^{2k+2}(x) \in (f(x), a_2)$. Then, since

$$f(f^{2k+1}(x)) < a_2 < f(f(x)),$$

we can find a $z_1 \in (f^{2k+1}(x), f(x))$ with $f(z_1) = a_2$. Furthermore, by the inequalities

$$f(f^2(x)) = f^3(x) < f(x) < f^{2k+2}(x) < a_2 = f(a_2),$$

there exist a $z_2 \in (a_2, f^2(x))$ and a $z_3 \in (f(x), a_1)$ such that $f(z_2) = f^{2k+2}(x)$ and $f(z_3) = z_2$. Thus

$$z_1 < a_2 = f^{n-2k-1}(a_2) = f^{n-2k-1}(f(z_1)) = f^{n-2k}(z_1)$$

and

$$f^{n-2k}(z_3) = f^{n-2k-2}(f^2(z_3)) = f^{n-2k-2}(f^{2k+2}(x)) = x < z_3,$$

which contradicts conditions (1) and (3). At last if $f^{2k+2}(x) \in (a_2, f^{2k}(x))$ then

$$f^{n-2}(f^{2k}(x)) = f^{2k-2}(x) < f^{2k}(x)$$

and

$$f^{2k+2}(x) < f^{2k}(x) = f^{n-2}(f^{2k+2}(x)),$$

contrary to (1). This completes the proof of Theorem B.

The proof of Theorem A consists of the following Lemmas 2–10.

Lemma 2. *If* $\mathrm{Per}(f, n) \neq \emptyset$ *for some* $n \in \mathbf{N}$ *then* $\mathrm{Per}(f, 1) \neq \emptyset$.

Lemma 3. *If* $\mathrm{Per}(f, n) \neq \emptyset$ *for some* $n \geq 2$ *then* $\mathrm{Per}(f, 2) \neq \emptyset$.

Proof. Fix an $x_0 \in \mathrm{Per}(f, n)$ and put $x = \min\{x_0, \dots, f^{n-1}(x_0)\}$. Let $y \in \{x_0, \dots, f^{n-1}(x_0)\}$ and $z \in [x, y)$ be such that $f(y) = x$ and $f(z) = y$. If $z = x$ then $x \in \mathrm{Per}(f, 2)$. Thus we may assume that $x < z$. Put

$$u = \sup\{v \in [x, z]: f^2(v) = v\}.$$

Then $u < z$. Suppose that $f(u) = u$ and choose an $a \in (z, y) \cap \mathrm{Per}(f, 1)$. Since

$$f(u) = u < a < y = f(z)$$

we have $a = f(b)$ for some $b \in (u, z)$. Thus $f^2(b) > b$ and $f^2(z) < z$ contrary to the definition of u. Consequently $u \in \mathrm{Per}(f, 2)$.

Lemma 4. *Let* $n \in \mathbf{N}$. *If* $\mathrm{Per}(f, 2^n) \neq \emptyset$ *then* $\mathrm{Per}(f, 2^k) \neq \emptyset$ *for any* $k \in \{0, \dots, n\}$.

Proof. It is enough to consider $k \in \{2, \dots, n\}$ only. By Lemma 1(iii) $\mathrm{Per}(f^{2^{k-1}}, 2^{n-k+1}) \neq \emptyset$. Using Lemma 3 and Lemma 1(iii) again we have

$$\emptyset \neq \mathrm{Per}(f^{2^{k-1}}, 2) = \mathrm{Per}(f, 2^k).$$

Lemma 5. *Let* $k \in \mathbf{N}_0$ *and let* $l \geq 3$ *be odd. If* $\mathrm{Per}(f, 2^k \cdot l) \neq \emptyset$ *then* $\mathrm{Per}(f, 2^n) \neq \emptyset$ *for any* $n \in \mathbf{N}_0$.

Proof. Use Lemma 1(ii) (taking $n = 2^k l$, $m = 2^{n+k}$), Lemma 3, and then Lemmas 1(iii) and 4.

Lemma 6. *If*

$$f^{2k}(x_0) < \cdots < f^2(x_0) < x_0 < f(x_0) < \cdots < f^{2k-1}(x_0), \qquad f^{2k+1}(x_0) \leq x_0$$

or

$$f^{2k}(x_0) > \cdots > f^2(x_0) > x_0 > f(x_0) > \cdots > f^{2k-1}(x_0), \qquad f^{2k+1}(x_0) \geq x_0$$

for some $x_0 \in I$ *and* $k \in \mathbf{N}$ *then* $\mathrm{Per}(f, 2k+1) \neq \emptyset$.

Proof. Let us assume that the first of the above conditions holds and fix an $a \in (x_0, f(x_0)) \cap \mathrm{Per}(f, 1)$. Let $I_0, \dots, I_{2k+1}$ be compact intervals such that

$I_{2k+1} = [x_0, f^{2k-1}(x_0)]$,

I_i is contained in the compact interval with the endpoints

$$f^i(x_0) \text{ and } f^{i-2}(x_0), \qquad i = 2k, \dots, 2,$$

$I_1 \subset [a, f(x_0)], \qquad I_0 \subset [x_0, a]$,

$f(I_i) = I_{i+1}, \qquad i = 2k, \dots, 0.$

Then $f^{2k+1}(I_0) = I_{2k+1} \supset I_0$ whence we can find a $c \in I_0 \cap \mathrm{Per}(f^{2k+1}, 1)$. Thus it follows from the relations

$$f^i(I_0) \cap I_0 \subset \{x_0\}, \qquad i = 2, \dots, 2k \quad \text{and} \quad f(I_0) \cap I_0 \subset \{a\}$$

that $c \in \mathrm{Per}(f, 2k+1)$.

Lemma 7. *Assume that there exists an* $x_0 \in I$ *such that*

$$f^2(x_0) < x_0 < f(x_0) \quad \textit{and} \quad f^3(x_0) \le a$$

or

$$f^2(x_0) > x_0 > f(x_0) \quad \textit{and} \quad f^3(x_0) \ge a$$

where $a \in \mathrm{Per}(f, 1)$ *belongs to the interval with the endpoints* x_0 *and* $f(x_0)$. *Then, for any* $k \in \mathbf{N}$, $\mathrm{Per}(f, 2k) \neq \varnothing$.

Proof. Assume the first of the above cases. Take $x_{-1} = f(x_0)$. Since

$$f(f(x_0)) < x_0 < a = f(a),$$

there is an $x_1 \in (a, f(x_0))$ such that $f(x_1) = x_0$. By the inequalities

$$f(a) = a < x_1 < f(x_0)$$

we can find an $x_2 \in (x_0, a)$ with $f(x_2) = x_1$. Continuing this procedure inductively we obtain a sequence $(x_n : n \in \mathbf{N})$ with the properties

$$x_{2n-2} < x_{2n} < a < x_{2n-1} < x_{2n-3} \quad \text{and} \quad f(x_n) = x_{n-1}, \qquad n \in \mathbf{N}.$$

Let $I_0, \dots, I_{2k}$ be compact intervals such that

$$I_{2k} = [f^3(x_0), f(x_0)], \qquad I_{2k-1} \subset [f^2(x_0), x_0],$$

I_i is contained in the compact interval with the endpoints

$$x_{2k-i-1} \text{ and } x_{2k-i-3}, \qquad i = 2k-2, \dots, 0,$$

$$f(I_i) = I_{i+1}, \qquad i = 2k-1, \dots, 0.$$

Then $f^{2k}(I_0) = I_{2k} \supset I_0$ whence there is a $c \in I_0 \cap \mathrm{Per}(f^{2k}, 1)$. Thus, due to the relations

$$f^i(I_0) \cap I_0 = I_i \cap I_0 = \varnothing, \qquad i \in \{1, \dots, 2k-1\} \backslash \{2\},$$

and (in the case $k > 1$)

$$f^2(I_0) \cap I_0 = I_2 \cap I_0 = \{x_{2k-3}\},$$

$c \in \mathrm{Per}(f, 2k)$ which was to be proved.

Lemma 8. *Let* $n \ge 3$ *be odd. If* $\mathrm{Per}(f, n) \neq \varnothing$ *then* $\mathrm{Per}(f, m) \neq \varnothing$ *for any* $m \in \mathbf{N}$ *such that* $n \dashv m$.

Proof. It is enough to consider the case where $\mathrm{Per}(f, k) = \varnothing$ for any $k \dashv n$. In view of Theorem B we may assume for example that

$$f^{n-1}(y_0) < \cdots < y_0 < f(y_0) < \cdots < f^{n-2}(y_0)$$

for a $y_0 \in \operatorname{Per}(f, n)$. If m is even then, taking $x_0 = f^{n-3}(y_0)$ in Lemma 7, we get the assertion. Assume that $m > n$ is odd. Proceeding analogously as in the proof of Lemma 7 we get the sequence $(y_i : i = 1, \dots, m-n)$ such that

$$y_0 < y_2 < \cdots < y_{m-n} < y_{m-n-1} < \cdots < y_1 < f(y_0)$$

and

$$f(y_i) = y_{i-1}, \qquad i = 1, \dots, m-n.$$

Now it is enough to put $x_0 = y_{m-n}$ and use Lemma 6.

Lemma 9. *Let* $n \geq 2$. *If* $\operatorname{Per}(f^2, n) \neq \varnothing$ *then* $\operatorname{Per}(f, 2n) \neq \varnothing$.

Proof. Let $x_0 \in \operatorname{Per}(f^2, n)$ and let $r = \operatorname{card}\{x_0, \dots, f^{2n-1}(x_0)\}$. Clearly $r \geq n$ and $r|2n$. Thus, $r \in \{n, 2n\}$. If n is even then $f^n(x_0) \neq x_0$ whence $r = 2n$ and $x_0 \in \operatorname{Per}(f, 2n)$. If n is odd and $r = n$ we may use Lemma 8.

Lemma 10. *Let* $n \in \mathbf{N}_0$ *and let* $l \geq 3$ *be odd. If* $\operatorname{Per}(f, 2^n l) \neq \varnothing$ *then* $\operatorname{Per}(f, 2^m k) \neq \varnothing$ *for* $m = n$ *and any odd* $k \geq l$ *and for any* $m > n$ *and odd* $k \geq 3$.

Proof. If $n = 0$ the assertion follows from Lemma 8. Fix an $n \in \mathbf{N}$ and assume that the lemma holds for $n - 1$. By Lemma 1(ii) $\operatorname{Per}(f^2, 2^{n-1} l) \neq \varnothing$. Hence, in view of the induction hypothesis,

$$\operatorname{Per}(f^2, 2^{m-1} k) \neq \varnothing, \qquad m > n \text{ and } k \geq 3 \quad \text{or} \quad m = n \text{ and } k \geq 1.$$

Now use Lemma 9.

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