PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 107, Number 1, September 1989

SOME REMARKS ON THE AVERAGE RANGE OF A VECTOR MEASURE

FRANCISCO J. FRENICHE

(Communicated by R. Daniel Mauldin)

ABSTRACT. We study some conditions on the average range of a vector measure with values in the bidual of a Banach space which imply that the range is contained in the space. We prove that Geitz's condition is a sufficient one if the dual closed unit ball is weak-star sequentially compact. We also show how to reduce to measures with values in the bidual of l^∞.

In this paper we consider the following question: given a vector measure ν with values in the bidual X^{**} of the Banach space X, under what conditions can we say that ν actually takes its values inside X? We shall also assume that ν is defined on a probability space (S, Σ, μ) and that it is absolutely continuous with respect to μ. This problem is connected with the following ones.

(a) Let (f_n) be an equi-integrable sequence of functions, bounded in L^1-norm; when is (f_n) weakly null in $L^1(\mu)$? This problem is equivalent to the former one when $X = c_0$ so that $X^{**} = l^\infty$. Indeed, it suffices to look at the vector measure given by

$$\nu(A) = \left(\int_A f_n \, d\mu\right)$$

and to apply Dunford-Pettis's characterization of the weakly null sequences in $L^1(\mu)$ [4].

(b) Let $f: S \to X$ be a bounded and scalarly measurable function. The indefinite Dunford integral ν of f is defined by

$$\langle \nu(A), x^* \rangle = \int_A \langle x^*, f(s) \rangle \, d\mu(s)$$

for $A \in \Sigma$ and $x^* \in X^*$; when is $\nu(A) \in X$ for any measurable A?, that is, when is f Pettis integrable? [3].

There are a lot of easy conditions which imply that the range of ν is contained in X. Let us assume for instance that for every $A \in \Sigma^+$—that is, $A \in \Sigma$ and $\mu(A) > 0$—there exists $B \in \Sigma_A^+$—that is, $B \in \Sigma^+$ and $B \subset A$—so that

Received by the editors July 25, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 28B05, 46G10.
Key words and phrases. Weakly convergent sequences in L^1, Pettis integral, core of a vector measure.

$\nu(B) \in X$; then an exhaustion argument gives $\nu(\Sigma) \subset X$. Another sufficient condition is the following one: $\nu(A) \in X$ for A running in a generating subalgebra of Σ; then the continuity of ν when regarded as a function on the pseudometric space associated to μ, as well as the density of the elements of that subalgebra, allow us to conclude that X contains the range of ν.

We are interested in conditions on the average range

$$\mathscr{A}_\nu(A) = \left\{ \frac{\nu(B)}{\mu(B)} : B \in \Sigma_A^+ \right\}$$

of the vector measure which guarantee that its range is contained in X. We recall Geitz's condition for Pettis integrability: Geitz defines in [5] the core of f on A by

$$\mathrm{core}_f(A) = \cap\{\text{weak-star closure of } \mathrm{co}\, f(A \setminus B) : \mu(B) = 0\}$$

which, as noticed in [7], can be expressed in terms of the average range of ν, the indefinite Dunford integral of f, namely

$$\mathrm{core}_\nu(A) = \text{weak-star closure of } \mathrm{co}(\mathscr{A}_\nu(A)).$$

The condition of Geitz reads as follows

$$\mathrm{core}_\nu(A) \cap X \neq \emptyset$$

for every $A \in \Sigma^+$.

The next theorem answers the question when the Banach space X has Mazur property, in particular, when X is a weakly compact generated Banach space [2]. Thus a sufficient condition for question (a) is Geitz's one.

Theorem 1. *Let $\nu : \Sigma \to X^{**}$ be an absolutely continuous vector measure satisfying Geitz's condition. Then $\nu(A)$ is weak-star sequentially continuous, for every $A \in \Sigma$.*

Proof. Assume, by contradiction, that there is $B \in \Sigma$ and a weak-star null sequence (x_n^*) so that $\langle \nu(B), x_n^* \rangle \geq 2$ for every n.

Let f_n be the Radon-Nikodym derivative of the scalar measure $x_n^* \circ \nu$ with respect to μ. As (x_n^*) is bounded in X and ν is absolutely continuous with respect to μ, the sequence (f_n) is equi-integrable and bounded in $L^1(\mu)$, so we can assume without lost of generality that (f_n) is weakly converging in $L^1(\mu)$ to a function f.

As the weak- and norm-closure are the same for convex sets, we can construct a sequence (J_n) with $J_n \subset \mathbf{N}$ finite, $\max J_n < \min J_{n+1}$, and a sequence (F_n) with $F_n \in \mathrm{co}(\{f_k : k \in J_n\})$ satisfying $\|F_n - f\|_1 \to 0$. By passing to a subsequence if necessary, we can also assume that (F_n) converges to f almost everywhere, hence by Egorov's theorem, that $F_n \to f$ almost uniformly on S.

On the other hand we can choose $B_1 \in \Sigma_B^+$ so that $f \geq 1$ on B_1 almost everywhere. Indeed, it suffices to notice that $\mu(S) = 1$ and $\int_B f\, d\mu \geq 2$.

Let $A \in \Sigma^+$ so that $A \subset B_1$ and $F_n \to f$ uniformly on A. Notice that we also have $f \geq 1$ almost everywhere on A.

We claim that $X \cap \mathrm{core}_\nu(A) = \varnothing$. To see this, let us fix $x \in X$. Then we can choose n so that $\langle x_k^*, x\rangle \le \frac{1}{6}$ for $k \ge n$, and m so that $\min J_m \ge n$ and $|F_m(s) - f(s)| \le \frac{1}{3}$, for every $s \in A$.

Let x^* be the convex combination of $\{x_k^*: k \in J_m\}$ constructed with the same weights as F_m when we write it in terms of $\{f_k: k \in J_m\}$. Then we have $\langle x^*, x\rangle \le \frac{1}{6}$.

At the same time,

$$\langle \nu(C), x^*\rangle = \int_C F_m \, d\mu \ge \int_C f \, d\mu - \frac{1}{3}\mu(C)$$

and so

$$\langle \nu(C), x^*\rangle \ge \tfrac{2}{3}\mu(C)$$

for every $C \in \Sigma_A^+$.

Therefore we have $x^* \ge \frac{2}{3}$ on $\mathscr{A}_\nu(A)$, so that $x \notin \mathrm{core}_\nu(A)$.

Our next theorem provides a complete answer for a certain class of Banach spaces. In its proof we use Theorem 1.

Theorem 2. *Let X be a Banach space such that its dual closed unit ball is weak-star sequentially compact. If the measure ν satisfies Geitz's condition, then its range is contained in X.*

Proof. By Grothendieck's completeness theorem, if $\nu(B) \notin X$ then there is $\varepsilon > 0$ such that for any $Y \subset X$ finite, there exists $x^* \in X^*$ with $\|x^*\| \le 1$, $|\langle x^*, x\rangle| \le 1$ for $x \in Y$, and $\langle \nu(B), x^*\rangle \ge \varepsilon$.

Let f_{x^*} be the Radon-Nikodym derivative of $x^* \circ \nu$ with respect to μ, and let

$$\mathscr{B}_Y = \{f_{x^*}: \|x^*\| \le 1, \langle \nu(B), x^*\rangle \ge \varepsilon \text{ and } |\langle x^*, x\rangle| \le 1 \text{ for } x \in Y\}.$$

By weak compactness, we can choose f in the weak closure of $\mathscr{B}_Y$ in $L^1(\mu)$, for every $Y \subset X$ finite. As the sets $\mathscr{B}_Y$ are convex, we get that f is in the norm closure of $\mathscr{B}_Y$ in $L^1(\mu)$, for every such Y.

Since $\int_B f \, d\mu \ge \varepsilon$ and $\mu(S) = 1$, we have $f \ge \varepsilon/2$ almost everywhere on some $A \in \Sigma_B^+$.

We now show that $\mathrm{core}_\nu(A)$ and X are disjoint sets, contradicting the hypothesis of the theorem. Let $x \in X$ be given.

Then $f \in \mathscr{B}_{\{nx\}}$ for every n, and so we can choose a sequence (x_n^*) which satisfies $\|x_n^*\| \le 1$, $\langle x_n^*, x\rangle \le n^{-1}$ and, if f_n is the density of $x_n^* \circ \nu$, (f_n) converges to f in $L^1(\mu)$.

The weak-star sequential compactness of the dual closed unit ball allows us to assume that the sequence (x_n^*) converges to some x^* in the weak-star topology.

From Theorem 1, for any $C \in \Sigma_A^+$,

$$\langle \nu(C), x^*\rangle = \lim \langle \nu(C), x_n^*\rangle$$

and, on the other hand,

$$\lim \left\langle \frac{\nu(C)}{\mu(C)}, x_n^* \right\rangle = \lim \frac{1}{\mu(C)} \int_C f_n \, d\mu$$

hence

$$\left\langle \frac{\nu(C)}{\mu(C)}, x^* \right\rangle \geq \varepsilon/2$$

and so $x^* \geq \varepsilon/2$ on $\mathscr{A}_\nu(A)$, despite the fact $\langle x^*, x\rangle \leq 0$.

Remark 1. The Banach space $\mathscr{C}([0, \omega_1])$, where ω_1 is the first uncountable ordinal, has weak-star sequentially compact dual closed unit ball, whereas the functional $\lambda \to \lambda(\{\omega_1\})$ is weak-star sequentially continuous on its dual space, and cannot be represented as a member of $\mathscr{C}([0, \omega_1])$.

So Theorem 1 does not include Theorem 2.

Remark 2. The proof of Theorem 2 can be changed to obtain the following result.

Let $X^\#$ be the subspace of X^{**} which consists of all those x^{**} satisfying the following property: for every bounded (x_n^*) and every weak-star cluster point x^* of (x_n^*), $\langle x^{**}, x^*\rangle$ is a cluster point of $(\langle x^{**}, x_n^*\rangle)$.

Then the range of ν is contained in X provided that it is contained in $X^\#$ and ν satisfies the condition of Geitz.

Moreover, in this case Geitz's condition can be weakened as in [7]: $\mathrm{core}_\nu(A) \cap \tilde{X} \neq \varnothing$ for any $A \in \Sigma^+$, where $\tilde{X}$ is the subspace of X^{**} which consists of those x^{**} which are weak-star cluster points of countable subsets of X. Indeed, to see that the core of the set A in the proof of Theorem 2 does not meet $\tilde{X}$, we notice first that, for any given weak-star cluster point x^{**} of some sequence (x_n), we can choose (x_n^*) with $\|x_n^*\| \leq 1$, $|\langle x_n^*, x_k\rangle| \leq n^{-1}$ if $k \leq n$, and such that the sequence of densities (f_n) converges to f in $L^1(\mu)$. Then we obtain that $\langle x^{**}, x^*\rangle = 0$ although $x^* \geq \varepsilon/2$ on $\mathscr{A}_\nu(A)$, if x^* is a weak-star cluster point of (x_n^*).

It is worthwhile to make some comments on the subspace $X^\#$.

(i) It is a closed subspace of X^{**} which contains the Hewitt realcompactification of X (this follows from Corson's theorem [1]). For any nonrealcompact Banach space X, $X^\#$ does not coincide either with X or with X^{**}, as a consequence of James's theorem [2].

(ii) When X has weak-star sequentially compact dual unit ball, weak-star sequentially continuous functionals on X^* belong to $X^\#$, hence Theorem 2 follows from this Remark and Theorem 1. Nevertheless, we do not know if weak Geitz's condition is enough in Theorem 1 and so in Theorem 2.

(iii) $X^\# \cap \tilde{X} = X$. For, given a countable $Y \subset X$ and x^{**} in the weak-star closure of Y, we define $f\colon X^* \to \mathbb{R}$ by

$$f(x^*) = \sum_1^\infty 2^{-n} |\langle x^*, x_n\rangle|$$

where (x_n) is an enumeration of Y. Then, if x^{**} were not in X, we could take (x_k^*) in the unit ball of X and $\delta > 0$ so that $f(x_k^*) < k^{-1}$ but $|\langle x^{**}, x_k^*\rangle| \geq \delta$. If $x^{**} \in X^{\#}$, then for some weak-star cluster point x^* of (x_k^*) we have $\langle x^{**}, x^*\rangle = 0$, which is a contradiction.

Remark 3. Modifying the proof of Theorem 2 we can prove Talagrand's theorem solving question (b) [8]. To see this we just need to extract one more subsequence from (x_n^*) so that f is the almost everywhere limit of (f_n). This proof is similar to that given in [6].

We finally show that a pathological measure with values in the bidual of l^∞ could be constructed from any pathological measure. By a pathological measure we mean here a vector measure which satisfies the assumptions of Theorem 1 but takes some of its values outside X.

Theorem 3. *Let ν be an X^{**}-valued vector measure, absolutely continuous with respect to the probability μ. If ν satisfies Geitz's condition but $\nu(\Sigma)$ is not contained in X, then there is a vector measure $\eta\colon \Sigma \to (l^\infty)^{**}$, which also satisfies Geitz's condition and whose range is not in l^∞.*

Proof. We proceed just as in Theorem 2, and choose $\varepsilon > 0$, f in the weak closure of the sets $\mathscr{B}_Y$ for any finite $Y \subset X$, $A \in \Sigma$ such that $f \geq \varepsilon/2$ almost everywhere on A.

We choose $x \in X \cap \operatorname{core}_\nu(A)$ and (x_n^*) in the unit ball of X^* such that $\langle x_n^*, x\rangle \leq n^{-1}$ and $f_n \to f$ in $L^1(\mu)$, if f_n is the density of $x_n^* \circ \nu$.

Let $T\colon X \to l^\infty$ be the bounded linear operator given by $T_y = (\langle x_n^*, y\rangle)$ for $y \in X$.

It is easy to check that the equation $\langle \eta(C), \lambda\rangle = \langle \nu(C), T^*\lambda\rangle$ for $\lambda \in (l^\infty)^*$, defines an absolutely continuous vector measure $\eta\colon \Sigma \to (l^\infty)^{**}$.

Furthermore, this vector measure satisfies Geitz's condition. For, if $C \in \Sigma^+$, we take $y \in X \cap \operatorname{core}_\nu(C)$, and we have $T_y \in \operatorname{core}_\eta(C)$.

Finally, let us suppose that the range of η lay in l^∞. Then we fix a non-principal ultrafilter on the positive integers and we denote by λ the member of the dual of l^∞ induced by it.

If (e_n) is the unit vector basis of l^1 regarded as members of l^∞, then we have $T^* e_n = x_n^*$.

Let $x^* = T^*\lambda$. Since λ is a weak-star cluster point of (e_n) and $\eta(C)$ is weak-star continuous, $\langle \eta(C), \lambda\rangle = \langle \nu(C), x^*\rangle$ is a cluster point of the scalar sequence $(\langle \nu(C), x_n^*\rangle)$.

Moreover, as T^* is weak-star to weak-star continuous, x^* is a weak-star cluster point of (x_n^*). It follows that $\langle x^*, x\rangle \leq 0$.

On the other hand, the average of ν on each $C \in \Sigma_A^+$ satisfies

$$\left\langle \frac{\nu(C)}{\mu(C)}, x^* \right\rangle \geq \varepsilon/2$$

contradicting the fact that $x \in \operatorname{core}_\nu(A)$.

REFERENCES

1. H. Corson, *The weak topology of a Banach space*, Trans. Amer. Math. Soc. **101** (1961), 1–15.
2. J. Diestel, *Geometry of Banach spaces—selected topics*, Lecture Notes in Mathematics, Vol. 485, Springer-Verlag, Berlin-Heidelberg-New York, 1974.
3. J. Diestel and J. J. Uhl, *Vector measures*, Math. Surveys **15**, Amer. Math. Soc., Providence, R.I., 1977.
4. N. Dunford and J. T. Schwartz, *Linear operators*. I: *General theory*, Interscience, New York, 1967.
5. R. F. Geitz, *Geometry and the Pettis integral*, Transl. Amer. Math. Soc. **269** (1982), 535–548.
6. R. Huff, *Remarks on Pettis integrability*, Proc. Amer. Math. Soc. **96** (1986), 402–404.
7. D. Sentilles and R. F. Wheeler, *Pettis integration via the Stonian transform*, Pacific J. Math. **107** (1983), 473–496.
8. M. Talagrand, *Pettis integral and measure theory*, Mem. Amer. Math. Soc., Providence, R.I., 1984.

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE SEVILLA, APTDO. 1160, 41080-SEVILLA, SPAIN