

SOME REMARKS ON THE AVERAGE RANGE OF A VECTOR MEASURE

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ABSTRACT. We study some conditions on the average range of a vector measure with values in the bidual of a Banach space which imply that the range is contained in the space. We prove that Geitz's condition is a sufficient one if the dual closed unit ball is weak-star sequentially compact. We also show how to reduce to measures with values in the bidual of l^∞ .

In this paper we consider the following question: given a vector measure ν with values in the bidual X^{**} of the Banach space X , under what conditions can we say that ν actually takes its values inside X ? We shall also assume that ν is defined on a probability space (S, Σ, μ) and that it is absolutely continuous with respect to μ . This problem is connected with the following ones.

(a) Let (f_n) be an equi-integrable sequence of functions, bounded in L^1 -norm; when is (f_n) weakly null in $L^1(\mu)$? This problem is equivalent to the former one when $X = c_0$ so that $X^{**} = l^\infty$. Indeed, it suffices to look at the vector measure given by

$$\nu(A) = \left(\int_A f_n d\mu \right)$$

and to apply Dunford-Pettis's characterization of the weakly null sequences in $L^1(\mu)$ [4].

(b) Let $f: S \rightarrow X$ be a bounded and scalarly measurable function. The indefinite Dunford integral ν of f is defined by

$$\langle \nu(A), x^* \rangle = \int_A \langle x^*, f(s) \rangle d\mu(s)$$

for $A \in \Sigma$ and $x^* \in X^*$; when is $\nu(A) \in X$ for any measurable A ?, that is, when is f Pettis integrable? [3].

There are a lot of easy conditions which imply that the range of ν is contained in X . Let us assume for instance that for every $A \in \Sigma^+$ —that is, $A \in \Sigma$ and $\mu(A) > 0$ —there exists $B \in \Sigma_A^+$ —that is, $B \in \Sigma^+$ and $B \subset A$ —so that

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$\nu(B) \in X$; then an exhaustion argument gives $\nu(\Sigma) \subset X$. Another sufficient condition is the following one: $\nu(A) \in X$ for A running in a generating subalgebra of Σ ; then the continuity of ν when regarded as a function on the pseudometric space associated to μ , as well as the density of the elements of that subalgebra, allow us to conclude that X contains the range of ν .

We are interested in conditions on the average range

$$\mathcal{A}_\nu(A) = \left\{ \frac{\nu(B)}{\mu(B)} : B \in \Sigma_A^+ \right\}$$

of the vector measure which guarantee that its range is contained in X . We recall Geitz's condition for Pettis integrability: Geitz defines in [5] the core of f on A by

$$\text{core}_f(A) = \cap \{ \text{weak-star closure of } \text{co } f(A \setminus B) : \mu(B) = 0 \}$$

which, as noticed in [7], can be expressed in terms of the average range of ν , the indefinite Dunford integral of f , namely

$$\text{core}_\nu(A) = \text{weak-star closure of } \text{co}(\mathcal{A}_\nu(A)).$$

The condition of Geitz reads as follows

$$\text{core}_\nu(A) \cap X \neq \emptyset$$

for every $A \in \Sigma^+$.

The next theorem answers the question when the Banach space X has Mazur property, in particular, when X is a weakly compact generated Banach space [2]. Thus a sufficient condition for question (a) is Geitz's one.

Theorem 1. *Let $\nu : \Sigma \rightarrow X^{**}$ be an absolutely continuous vector measure satisfying Geitz's condition. Then $\nu(A)$ is weak-star sequentially continuous, for every $A \in \Sigma$.*

Proof. Assume, by contradiction, that there is $B \in \Sigma$ and a weak-star null sequence (x_n^*) so that $\langle \nu(B), x_n^* \rangle \geq 2$ for every n .

Let f_n be the Radon-Nikodym derivative of the scalar measure $x_n^* \circ \nu$ with respect to μ . As (x_n^*) is bounded in X and ν is absolutely continuous with respect to μ , the sequence (f_n) is equi-integrable and bounded in $L^1(\mu)$, so we can assume without loss of generality that (f_n) is weakly converging in $L^1(\mu)$ to a function f .

As the weak- and norm-closure are the same for convex sets, we can construct a sequence (J_n) with $J_n \subset \mathbb{N}$ finite, $\max J_n < \min J_{n+1}$, and a sequence (F_n) with $F_n \in \text{co}(\{f_k : k \in J_n\})$ satisfying $\|F_n - f\|_1 \rightarrow 0$. By passing to a subsequence if necessary, we can also assume that (F_n) converges to f almost everywhere, hence by Egorov's theorem, that $F_n \rightarrow f$ almost uniformly on S .

On the other hand we can choose $B_1 \in \Sigma_B^+$ so that $f \geq 1$ on B_1 almost everywhere. Indeed, it suffices to notice that $\mu(S) = 1$ and $\int_B f d\mu \geq 2$.

Let $A \in \Sigma^+$ so that $A \subset B_1$ and $F_n \rightarrow f$ uniformly on A . Notice that we also have $f \geq 1$ almost everywhere on A .

We claim that $X \cap \text{core}_\nu(A) = \emptyset$. To see this, let us fix $x \in X$. Then we can choose n so that $\langle x_k^*, x \rangle \leq \frac{1}{6}$ for $k \geq n$, and m so that $\min J_m \geq n$ and $|F_m(s) - f(s)| \leq \frac{1}{3}$, for every $s \in A$.

Let x^* be the convex combination of $\{x_k^* : k \in J_m\}$ constructed with the same weights as F_m when we write it in terms of $\{f_k : k \in J_m\}$. Then we have $\langle x^*, x \rangle \leq \frac{1}{6}$.

At the same time,

$$\langle \nu(C), x^* \rangle = \int_C F_m d\mu \geq \int_C f d\mu - \frac{1}{3}\mu(C)$$

and so

$$\langle \nu(C), x^* \rangle \geq \frac{2}{3}\mu(C)$$

for every $C \in \Sigma_A^+$.

Therefore we have $x^* \geq \frac{2}{3}$ on $\mathcal{A}_\nu(A)$, so that $x \notin \text{core}_\nu(A)$.

Our next theorem provides a complete answer for a certain class of Banach spaces. In its proof we use Theorem 1.

Theorem 2. *Let X be a Banach space such that its dual closed unit ball is weak-star sequentially compact. If the measure ν satisfies Geitz's condition, then its range is contained in X .*

Proof. By Grothendieck's completeness theorem, if $\nu(B) \notin X$ then there is $\varepsilon > 0$ such that for any $Y \subset X$ finite, there exists $x^* \in X^*$ with $\|x^*\| \leq 1$, $|\langle x^*, x \rangle| \leq 1$ for $x \in Y$, and $\langle \nu(B), x^* \rangle \geq \varepsilon$.

Let f_{x^*} be the Radon-Nikodym derivative of $x^* \circ \nu$ with respect to μ , and let

$$\mathcal{B}_Y = \{f_{x^*} : \|x^*\| \leq 1, \langle \nu(B), x^* \rangle \geq \varepsilon \text{ and } |\langle x^*, x \rangle| \leq 1 \text{ for } x \in Y\}.$$

By weak compactness, we can choose f in the weak closure of \mathcal{B}_Y in $L^1(\mu)$, for every $Y \subset X$ finite. As the sets \mathcal{B}_Y are convex, we get that f is in the norm closure of \mathcal{B}_Y in $L^1(\mu)$, for every such Y .

Since $\int_B f d\mu \geq \varepsilon$ and $\mu(S) = 1$, we have $f \geq \varepsilon/2$ almost everywhere on some $A \in \Sigma_B^+$.

We now show that $\text{core}_\nu(A)$ and X are disjoint sets, contradicting the hypothesis of the theorem. Let $x \in X$ be given.

Then $f \in \mathcal{B}_{\{nx\}}$ for every n , and so we can choose a sequence (x_n^*) which satisfies $\|x_n^*\| \leq 1$, $\langle x_n^*, x \rangle \leq n^{-1}$ and, if f_n is the density of $x_n^* \circ \nu$, (f_n) converges to f in $L^1(\mu)$.

The weak-star sequential compactness of the dual closed unit ball allows us to assume that the sequence (x_n^*) converges to some x^* in the weak-star topology.

From Theorem 1, for any $C \in \Sigma_A^+$,

$$\langle \nu(C), x^* \rangle = \lim \langle \nu(C), x_n^* \rangle$$

and, on the other hand,

$$\lim \left\langle \frac{\nu(C)}{\mu(C)}, x_n^* \right\rangle = \lim \frac{1}{\mu(C)} \int_C f_n d\mu$$

hence

$$\left\langle \frac{\nu(C)}{\mu(C)}, x^* \right\rangle \geq \varepsilon/2$$

and so $x^* \geq \varepsilon/2$ on $\mathcal{A}_\nu(A)$, despite the fact $\langle x^*, x \rangle \leq 0$.

Remark 1. The Banach space $\mathcal{E}([0, \omega_1])$, where ω_1 is the first uncountable ordinal, has weak-star sequentially compact dual closed unit ball, whereas the functional $\lambda \rightarrow \lambda(\{\omega_1\})$ is weak-star sequentially continuous on its dual space, and cannot be represented as a member of $\mathcal{E}([0, \omega_1])$.

So Theorem 1 does not include Theorem 2.

Remark 2. The proof of Theorem 2 can be changed to obtain the following result.

Let $X^\#$ be the subspace of X^{**} which consists of all those x^{**} satisfying the following property: for every bounded (x_n^*) and every weak-star cluster point x^* of (x_n^*) , $\langle x^{**}, x^* \rangle$ is a cluster point of $(\langle x^{**}, x_n^* \rangle)$.

Then the range of ν is contained in X provided that it is contained in $X^\#$ and ν satisfies the condition of Geitz.

Moreover, in this case Geitz's condition can be weakened as in [7]: $\text{core}_\nu(A) \cap \tilde{X} \neq \emptyset$ for any $A \in \Sigma^+$, where \tilde{X} is the subspace of X^{**} which consists of those x^{**} which are weak-star cluster points of countable subsets of X . Indeed, to see that the core of the set A in the proof of Theorem 2 does not meet \tilde{X} , we notice first that, for any given weak-star cluster point x^{**} of some sequence (x_n) , we can choose (x_n^*) with $\|x_n^*\| \leq 1$, $|\langle x_n^*, x_k \rangle| \leq n^{-1}$ if $k \leq n$, and such that the sequence of densities (f_n) converges to f in $L^1(\mu)$. Then we obtain that $\langle x^{**}, x^* \rangle = 0$ although $x^* \geq \varepsilon/2$ on $\mathcal{A}_\nu(A)$, if x^* is a weak-star cluster point of (x_n^*) .

It is worthwhile to make some comments on the subspace $X^\#$.

(i) It is a closed subspace of X^{**} which contains the Hewitt realcompactification of X (this follows from Corson's theorem [1]). For any nonrealcompact Banach space X , $X^\#$ does not coincide either with X or with X^{**} , as a consequence of James's theorem [2].

(ii) When X has weak-star sequentially compact dual unit ball, weak-star sequentially continuous functionals on X^* belong to $X^\#$, hence Theorem 2 follows from this Remark and Theorem 1. Nevertheless, we do not know if weak Geitz's condition is enough in Theorem 1 and so in Theorem 2.

(iii) $X^\# \cap \tilde{X} = X$. For, given a countable $Y \subset X$ and x^{**} in the weak-star closure of Y , we define $f: X^* \rightarrow \mathbb{R}$ by

$$f(x^*) = \sum_1^\infty 2^{-n} |\langle x^*, x_n \rangle|$$

where (x_n) is an enumeration of Y . Then, if x^{**} were not in X , we could take (x_k^*) in the unit ball of X and $\delta > 0$ so that $f(x_k^*) < k^{-1}$ but $|\langle x^{**}, x_k^* \rangle| \geq \delta$. If $x^{**} \in X^\#$, then for some weak-star cluster point x^* of (x_k^*) we have $\langle x^{**}, x^* \rangle = 0$, which is a contradiction.

Remark 3. Modifying the proof of Theorem 2 we can prove Talagrand’s theorem solving question (b) [8]. To see this we just need to extract one more subsequence from (x_n^*) so that f is the almost everywhere limit of (f_n) . This proof is similar to that given in [6].

We finally show that a pathological measure with values in the bidual of l^∞ could be constructed from any pathological measure. By a pathological measure we mean here a vector measure which satisfies the assumptions of Theorem 1 but takes some of its values outside X .

Theorem 3. *Let ν be an X^{**} -valued vector measure, absolutely continuous with respect to the probability μ . If ν satisfies Geitz’s condition but $\nu(\Sigma)$ is not contained in X , then there is a vector measure $\eta: \Sigma \rightarrow (l^\infty)^{**}$, which also satisfies Geitz’s condition and whose range is not in l^∞ .*

Proof. We proceed just as in Theorem 2, and choose $\varepsilon > 0$, f in the weak closure of the sets \mathcal{B}_Y for any finite $Y \subset X$, $A \in \Sigma$ such that $f \geq \varepsilon/2$ almost everywhere on A .

We choose $x \in X \cap \text{core}_\nu(A)$ and (x_n^*) in the unit ball of X^* such that $\langle x_n^*, x \rangle \leq n^{-1}$ and $f_n \rightarrow f$ in $L^1(\mu)$, if f_n is the density of $x_n^* \circ \nu$.

Let $T: X \rightarrow l^\infty$ be the bounded linear operator given by $T_y = (\langle x_n^*, y \rangle)$ for $y \in X$.

It is easy to check that the equation $\langle \eta(C), \lambda \rangle = \langle \nu(C), T^* \lambda \rangle$ for $\lambda \in (l^\infty)^*$, defines an absolutely continuous vector measure $\eta: \Sigma \rightarrow (l^\infty)^{**}$.

Furthermore, this vector measure satisfies Geitz’s condition. For, if $C \in \Sigma^+$, we take $y \in X \cap \text{core}_\nu(C)$, and we have $T_y \in \text{core}_\eta(C)$.

Finally, let us suppose that the range of η lay in l^∞ . Then we fix a non-principal ultrafilter on the positive integers and we denote by λ the member of the dual of l^∞ induced by it.

If (e_n) is the unit vector basis of l^1 regarded as members of l^∞ , then we have $T^* e_n = x_n^*$.

Let $x^* = T^* \lambda$. Since λ is a weak-star cluster point of (e_n) and $\eta(C)$ is weak-star continuous, $\langle \eta(C), \lambda \rangle = \langle \nu(C), x^* \rangle$ is a cluster point of the scalar sequence $(\langle \nu(C), x_n^* \rangle)$.

Moreover, as T^* is weak-star to weak-star continuous, x^* is a weak-star cluster point of (x_n^*) . It follows that $\langle x^*, x \rangle \leq 0$.

On the other hand, the average of ν on each $C \in \Sigma_A^+$ satisfies

$$\left\langle \frac{\nu(C)}{\mu(C)}, x^* \right\rangle \geq \varepsilon/2$$

contradicting the fact that $x \in \text{core}_\nu(A)$.

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