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INJECTIVE HULLS OF SIMPLE sl(2, C) MODULES ARE LOCALLY ARTINIAN

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ABSTRACT. Let L denote the simple Lie algebra $\mathrm{sl}(2, \mathbf{C})$ over the complex numbers $\mathbf{C}$. For any simple L-modules S, considered as a left unital module over the universal enveloping algebra of L, $U(L)$, the injective hull of S, $E_L(S)$, is a locally Artinian $U(L)$-module.

In this paper a module will mean unital left module over the universal enveloping algebra $U(L)$ of L and will be referred to simply as an L-module. A two-sided ideal will be referred to as an ideal. We recall that the standard ordered basis for L is

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with the relations $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$. It is well-known that the Casimir element $Q = 4ef + h^2 - 2h$ lies in the center $Z(L)$ of $U(L)$ and that $Z(L) = \mathbf{C}[Q]$. Furthermore, if I is a minimal primitive ideal of $U(L)$ then $I = U(L)(Q - c)$ for some complex number c. [3, 4.9.22, p. 167].

We will need some facts about primitive factor rings of $U(L)$ which we record here as lemmas for our convenience. If R is a ring, the Krull dimension of R will be denoted by $K - \dim(R)$.

Lemma 1. *Let R be a non-Artinian primitive factor ring of $U(L)$. Then $K - \dim(R) = 1$.*

Proof. The case where R is simple follows from the work of Arnal-Pinczon [1] and Roos [6]. If R is not simple, the result was proven by S. P. Smith [7].

Lemma 2 [5, Proposition 5.5, p. 465]. *Let R be a prime ring with $K - \dim(R) = n$. Then either R is simple Artinian or $K - \dim(W) < n$ whenever W is a finitely generated essential extension of a simple R-module.*

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Lemma 3 [4, Lemma 2, p. 131]. *Let* z *be a centralizing element in a left Noetherian ring* U *and* E *be a left* U*-module such that* $\operatorname{ann}_E(Uz) = \{e \in E \mid (Uz)e = 0\}$ *is Artinian and essential in* E. *Then* E *is an Artinian* U*-module.*

We are now prepared to prove our principal result.

Theorem. *Let* S *be a simple* $U(L)$*-module. Then the injective hull of* S *is locally Artinian.*

Proof. Let E be a finitely generated essential extension of S. It suffices to show that E is Artinian. Let I be the minimal primitive ideal which annihilates S. Then $I = Uz$ for some central element z in $U = U(L)$. By Lemma 1, $K-\dim(U/I) = 1$. Let $M = \{m \in E \mid Im = 0\}$. Then M is a finitely generated (U/I)-module and S is an essential submodule of M. Hence by Lemma 2, M is Artinian. Thus M is an essential Artinian submodule of E, so E is Artinian by Lemma 3.

Corollary. *Let* V *be an Artinian* L*-module. Then* $E_L(V)$ *is locally Artinian.*

Proof. Since $U(L)$ is Noetherian we may express $E_L(V)$ as a direct sum of indecomposable injective modules. Each summand must intersect V non-trivially so contains an Artinian submodule and hence contains a simple submodule which must then be an essential submodule of the summand. Thus each summand is the injective hull of a simple module so is locally Artinian by the Theorem. But a direct sum of locally Artinian modules is again locally Artinian, hence $E_L(V)$ is locally Artinian.

Remark 1. If L is a solvable Lie algebra over a field k of characteristic zero, then $E_L(V)$ is locally finite dimensional when V is a locally finite dimensional L-module [2, Corollary 12, p. 467]. If L is a semisimple Lie algebra over k then $E_L(V)$ cannot be locally finite dimensional in view of Weyl's Theorem on complete reducibility since any finite dimensional submodule $\neq V$ would contain V as a direct summand by complete reducibility.

Remark 2. If $M(\lambda)$ is a Verma module with a finite dimensional quotient $L(\lambda)$, then its $\mathfrak{h}$-finite dual $\delta M(\lambda)$ is an essential extension of $L(\lambda)$. As one allows $\mathfrak{h}$ to vary according to automorphisms of $U(\mathrm{sl}(2, \mathbf{C}))$, and twists $\delta M(\lambda)$ according to the automorphism, one gets further essential extensions of $L(\lambda)$. Note that twisting $L(\lambda)$ by an automorphism leaves $L(\lambda)$ unchanged up to isomorphism since it is completely determined by its dimension.

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REFERENCES

1. D. Arnal and G. Pinczon, *Idéaux à gauche dans les quotients simples de l'algèbre enveloppante de* sl(2), Bull. Soc. Math. France **101** (1973), 381–395.
2. R. P. Dahlberg, *Injective hulls of Lie modules*, J. Algebra **87**, No. 2 (1984), 458–471.
3. J. Dixmier, *Enveloping algebras*, North-Holland, Amsterdam, 1977.
4. A. V. Jategaonkar, *Certain injectives are Artinian*, Proc. Kent State Conf., Lect. Notes in Math. No. 545, Springer-Verlag, Berlin, 1975.
5. I. M. Musson, *Injective modules for group rings of polycyclic groups* II, Quart. J. Math. Oxford (2) **31** (1980), 449–466.
6. J. E. Roos, *Compléments a l'étude des quotients primitifs des algèbres enveloppantes des algèbres de Lie semi-simples*, C. R. Acad. Sci. Paris Sér. A **276** (1973), 447–450.
7. S. P. Smith, *The primitive factor rings of the enveloping algebra of* sl(2, **C**), J. London Math. Soc. (2) **24** (1981), 97–108.

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