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A NOTE ON THE DIOPHANTINE EQUATION $x^{2p} - Dy^2 = 1$

LE MAOHUA

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ABSTRACT. Let D be a positive integer which is square free, and let p be a prime. In this note we prove that if $p = 2$ and $D > \exp 64$, then the equation $x^{2p} - Dy^2 = 1$ has at most one positive integer solution (x, y); if $p > 2$ and $D > \exp\exp\exp\exp 10$, then every positive integer solution (x, y) can be expressed as $x^p + y\sqrt{D} = \varepsilon_1^m$, where m is a positive integer with $2 \nmid m$, ε_1 is the fundamental solution of Pell's equation $u^2 - Dv^2 = 1$.

I. INTRODUCTION

Let D be a positive integer which is square free, $\varepsilon_1 = u_1 + v_1\sqrt{D}$ be the fundamental solution of the equation

$$u^2 - Dv^2 = 1, \tag{1}$$

and let p be a prime. In 1942, Ljunggren [8] showed that if $p = 2$ then the equation

$$x^{2p} - Dy^2 = 1 \tag{2}$$

has at most two positive integer solutions. The author [5] proved that if $p = 2$ then (2) has positive integer solution (x, y) if and only if either u_1 or $u_1^2 + Dv_1^2$ is a square. In this note we prove:

Theorem 1. *If $p = 2$ and $D > \exp 64$, then (2) has at most one positive integer solution (x, y).* □

Clearly, every solution (x, y) of (2) can be expressed as

$$x^p + y\sqrt{D} = \varepsilon_1^m, \tag{3}$$

where m is a positive integer. Recently, Cao [3] showed that if $p > 2$ then $4 \nmid m$. We prove:

Theorem 2. *If $p > 2$ and $D > \exp\exp\exp\exp 10$, then $2 \nmid m$.* □

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2. Proof of Theorem 1

Lemma 1 [7, Formula 1.76]. *Let η be a positive integer and let α, β be complex numbers. Then*

$$\alpha^n + \beta^n = \sum_{i=0}^{[n/2]} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix} (\alpha+\beta)^{n-2i} (\alpha\beta)^i ,$$

where

$$\begin{bmatrix} n \\ i \end{bmatrix} \frac{(n-i-1)!n}{(n-2i)!i!}, \qquad i = 0, \dots, \binom{n}{2}$$

are positive integers. □

Lemma 2. *Let d be a positive integer which is nonsquare. If (U, V) is a positive integer solution of*

$$U^2 - dV^2 = \lambda, \qquad \lambda = \pm 1, \tag{4}$$

then every positive integer solution (U', V') of (4) such that $U|U'$ can be expressed as

$$U' + V'\sqrt{d} = (U + V\sqrt{d})^t, \tag{5}$$

where t is a positive integer with $2 \nmid t$.

Proof. Clearly, if U', V' satisfy (5), then (U', V') is a positive integer solution of (4) with $U|U'$. On the other hand, if (U, V) and (U', V') are positive integer solutions of (4) with $U \le U'$, then there exist positive integers s_1, s_2 such that

$$U = \frac{\rho^{s_1} + \overline{\rho}^{s_1}}{2}, \qquad U' = \frac{\rho^{s_2} + \overline{\rho}^{s_2}}{2}, \qquad s_1 \le s_2,$$

where $\rho = U_1 + V_1\sqrt{d}$ is the fundamental solution of (4). If $s_1 = s_2$, then $t = 1$ and the lemma holds. If $s_1 < s_2$, then there exist integers a, b such that $s_2 = 2as_1 \pm b$, $a > 0$, $0 \le b \le s_1$. From

$$\frac{\rho^{s_2} + \overline{\rho}^{s_2}}{2} + (\rho\overline{\rho})^{s_1} \frac{\rho^{s_2-2s_1} + \overline{\rho}^{s_2-2s_1}}{2} = 2\left(\frac{\rho^{s_1} + \overline{\rho}^{s_1}}{2}\right)\left(\frac{\rho^{s_2-s_1} + \overline{\rho}^{s_2-s_1}}{2}\right),$$

we see that if $U|U'$ then

$$U' \equiv \left|\frac{\rho^{s_2} + \overline{\rho}^{s_2}}{2}\right| \equiv \left|\frac{\rho^{s_2-2s_1} + \overline{\rho}^{s_2-2s_1}}{2}\right| \equiv \cdots \equiv \left|\frac{\rho^{s_2-2as_1} + \overline{\rho}^{s_2-2as_1}}{2}\right|$$

$$\equiv \left|\frac{\rho^{b} + \overline{\rho}^{b}}{2}\right| \equiv 0 \pmod{U}.$$

Since, if $b < s_1$ then $0 < \frac{1}{2}(\rho^b + \overline{\rho}^b) < \frac{1}{2}(\rho^{s_1} + \overline{\rho}^{s_1}) = U$. It follows that $b = s_1$ and $s_2 = (2a \pm 1)s_1$. Hence the lemma. □

Lemma 3. *If* (u, v) *and* (u', v') *are positive integer solutions of* (1) *with* $u | u'$, *then there exist fixed positive integers* D_1, D_2 *such that* $D_1 D_2 = D$,

$$
\begin{aligned}
&u + 1 = \delta D_1 v_1^2, \qquad u - 1 = \delta D_2 v_2^2, \\
&u' + 1 = \delta D_1 v_1'^2, \qquad u' - 1 = \delta D_2 v_2'^2,
\end{aligned} \tag{6}
$$

where δ, v_1, v_2, v_1', v_2' *are positive integers satisfy*

$$\delta = \begin{cases} 1, & 2|u, \\ 2, & 2 \nmid u, \end{cases} \qquad \delta v_1 v_2 = v, \ \delta v_1' v_2' = v'.$$

Proof. Since D is square free, from (1) we get (6) clearly. Let $\eta = v_1\sqrt{D_1} + v_2\sqrt{D_2}$, $\overline{\eta} = v_1\sqrt{D_1} - v_2\sqrt{D_2}$, then $\eta + \overline{\eta} = 2v_1\sqrt{D_1}$, $\eta - \overline{\eta} = 2v_2\sqrt{D_2}$, $\eta\overline{\eta} = 2/\delta$. Further let $\varepsilon = u + v\sqrt{D}$, $\overline{\varepsilon} = u - v\sqrt{D}$, then $\varepsilon = (\delta/2)\eta^2$, $\overline{\varepsilon} = (\delta/2)\overline{\eta}^2$. By Lemma 2, if $u|u'$ then

$$u' + v'\sqrt{D} = \varepsilon^t, \qquad u' - v'\sqrt{D} = \overline{\varepsilon}^t, \qquad t > 0, \ 2 \nmid t.$$

Clearly, by (6), the lemma holds when $t = 1$. If $t > 1$, by Lemma 1, then we have

$$
\begin{aligned}
u' + 1 &= \frac{\varepsilon^t + \overline{\varepsilon}^t}{2} + 1 = \frac{1}{2}\left(\left(\frac{\delta}{2}\eta^2\right)^t + \left(\frac{\delta}{2}\overline{\eta}^2\right)^t + 2\right) = \frac{1}{2}\left(\left(\frac{\delta}{2}\right)^t (\eta^t + \overline{\eta}^t)^2\right) \\
&= \delta D_1 v_1^2 \left(\sum_{i=0}^{(t-1)/2} (-1)^i \begin{bmatrix} t \\ i \end{bmatrix} (2\delta D_1 v_1^2)^{(t-1)/2-i}\right)^2
\end{aligned}
$$

and

$$u' - 1 = \delta D_2 v_2^2 \left(\sum_{i=0}^{(t-1)/2} \begin{bmatrix} t \\ i \end{bmatrix} (2\delta D_2 v_2^2)^{(t-1)/2-i}\right)^2.$$

The lemma is proved. □

Lemma 4 [4, Theorem 2]. *If* $p = 2$, *then* m *in* (3) *satisfies* $4 \nmid m$. □

Lemma 5 [5, Proof of Theorem]. *If* (u, v) *is a positive integer solution of* (1), $\varepsilon = u + v\sqrt{D}$, $\overline{\varepsilon} = u - v\sqrt{D}$, *then for any odd prime* q *and integer* z, $(\varepsilon^q + \overline{\varepsilon}^q)/(\varepsilon + \overline{\varepsilon}) = qz^2$ *is impossible.* □

Lemma 6 [10]. *Let* β, α_1, α_2 *be nonzero algebraic numbers with degrees* D_0, D_1, D_2 *and heights* H_0, H_1, H_2, *respectively,* $B = \max(e^{D_0}, H_0)$, $A_j = \max(e^{|\log\alpha_j|}, H_j)$ $(j = 1, 2)$, *and let* D *denote the degree of the field* $\mathbb{Q}(\beta, \alpha_1, \alpha_2)$. *If* $\Lambda = \beta\log\alpha_1 - \log\alpha_2 \neq 0$, *then*

$$|\Lambda| > \exp\left(-5 \cdot 10^8 D^4 \frac{S_1 S_2}{D_1 D_2} T^2 (\log E)^{-3}\right),$$

where $S_0 = D_0 + \log B$, $S_j = D_j + \log A_j$ $(j = 1, 2)$,

$$T = 4 + \frac{S_0}{D_0} + \log\left(D^2 \frac{S_1 S_2}{D_1 D_2}\right), \qquad E > e.$$

Proof of Theorem 1. When $p=2$, if (2) has two positive integer solutions (x_1, y_1) and (x_2, y_2) with $x_1 < x_2$, then there exist integers s_1, s_2 such that

$$x_1^2 = \frac{\varepsilon_1^{s_1} + \bar{\varepsilon}_1^{s_1}}{2}, \qquad x_2^2 = \frac{\varepsilon_1^{s_2} + \bar{\varepsilon}_1^{s_2}}{2}, \qquad 0 < s_1 < s_2, \tag{7}$$

where $\bar{\varepsilon}_1 = u_1 - v_1\sqrt{D}$. Write $s_2 = 2^r s'$, $r \geq 0$, $2 \nmid s'$. By Lemma 4, if $s' = 1$ then $r = 1$, thereby $s_1 = 1$, $s_2 = 2$, and $x_2^2 - 2x_1^4 = -1$, $x_2 > 1$, $x_1 > 1$. By [9], we see that $(x_1, x_2) = (13, 239)$ and $D = 1785$. It follows that if $D \neq 1785$, then $s' > 1$ and s' has an odd prime factor q. Put $\varepsilon = \varepsilon_1^{s_2/q}$, $\bar{\varepsilon} = \bar{\varepsilon}_1^{s_2/q}$, then $\varepsilon = u + v\sqrt{D}$, $\bar{\varepsilon} = u - v\sqrt{D}$, where (u, v) is a positive integer solution of (1). By Lemma 1, from (7) we get

$$x_2^2 = \frac{\varepsilon^q + \bar{\varepsilon}^q}{2} = \left(\frac{\varepsilon + \bar{\varepsilon}}{2}\right)\left(\frac{\varepsilon^q + \bar{\varepsilon}^q}{\varepsilon + \bar{\varepsilon}}\right) = u\left(\frac{\varepsilon^q + \bar{\varepsilon}^q}{\varepsilon + \bar{\varepsilon}}\right). \tag{8}$$

Since

$$\frac{\varepsilon^q + \bar{\varepsilon}^q}{\varepsilon + \bar{\varepsilon}} = \sum_{i=0}^{(q-1)/2} (-1)^i \begin{bmatrix} q \\ i \end{bmatrix} (2u)^{q-2i-1} \equiv (-1)^{(q-1)/2} q \pmod{u},$$

from (8) we obtain

$$u = cx_{21}^2, \qquad \frac{\varepsilon^q + \bar{\varepsilon}^q}{\varepsilon + \bar{\varepsilon}} = cx_{22}^2, \qquad c = 1 \text{ or } q, \qquad cx_{21}x_{22} = x_2. \tag{9}$$

Further, by Lemma 5, (9) is impossible when $c = q$. Therefore $c = 1$ and $u = x_{21}^2$. Note that (2) has at most two solutions when $p = 2$. We see that $x_{21} = x_1$ and $x_1 | x_2$. Then, by Lemma 3, we have

$$x_1^2 + 1 = \delta D_1 y_{11}^2, \qquad x_1^2 - 1 = \delta D_2 y_{12}^2, \tag{10}$$

$$x_2^2 + 1 = \delta D_1 y_{21}^2, \qquad x_2^2 - 1 = \delta D_2 y_{22}^2, \tag{11}$$

where

$$\delta = \begin{cases} 1, & 2|x_1, \\ 2, & 2 \nmid x_1, \end{cases} \qquad D_1 D_2 = D, \quad \delta y_{11} y_{12} = y_1, \quad \delta y_{21} y_{22} = y_2.$$

Let $\rho_1 = x_1 + y_{11}\sqrt{\delta D_1}$, $\bar{\rho}_1 = x_1 - y_{11}\sqrt{\delta D_1}$, $\rho_2 = x_1 + y_{12}\sqrt{\delta D_2}$, $\bar{\rho}_2 = x_1 - y_{12}\sqrt{\delta D_2}$. Since $x_1 < x_2$ and $x_1 | x_2$, by Lemma 2, from (10) and (11) we obtain

$$x_2 + y_{21}\sqrt{\delta D_1} = \rho_1^{t_1}, \qquad x_2 - y_{21}\sqrt{\delta D_1} = \bar{\rho}_1^{t_1}, \qquad t_1 > 1, \quad 2 \nmid t_1,$$

$$x_2 + y_{22}\sqrt{\delta D_2} = \rho_2^{t_2}, \qquad x_2 - y_{22}\sqrt{\delta D_2} = \bar{\rho}_2^{t_2}, \qquad t_2 > 1, \quad 2 \nmid t_2.$$

Hence

$$\rho_1 + \bar{\rho}_1 = \rho_2 + \bar{\rho}_2, \tag{12}$$

$$\rho_1^{t_1} + \bar{\rho}_1^{t_1} = \rho_2^{t_2} + \bar{\rho}_2^{t_2}. \tag{13}$$

Since $\rho_1\overline{\rho}_1 = -1$, $\rho_1 > 0$, $\overline{\rho}_1 < 0$, $\rho_2\overline{\rho}_2 = 1$, $\rho_2 > 0$, $\overline{\rho}_2 > 0$. From (12), (13) we get $\rho_1 > \rho_2$,

$$t_1 < t_2, \tag{14}$$

$$\log \rho_1 = \log \rho_2 + \Delta_1, \tag{15}$$

$$t_1 \log \rho_1 = t_2 \log \rho_2 + \Delta_2, \tag{16}$$

where

$$0 < \Delta_1 = \frac{2}{\rho_1\rho_2}\sum_{l=0}^{\infty}\frac{(\rho_1\rho_2)^{-2l}}{2l+1} < \frac{4}{\rho_1^2}, \qquad 0 < \Delta_2 = \frac{2}{\rho_1^{t_1}\rho_2^{t_2}}\sum_{l=0}^{\infty}\frac{(\rho_1^{t_1}\rho_2^{t_2})^{-2l}}{2l+1} < \frac{4}{\rho_2^{2t_2}}.$$

From (15), (16) we get

$$t_2 = \frac{(t_2 - t_1)\log \rho_1 + \Delta_2}{\Delta_1} > \frac{\rho_1^2}{2}\log \rho_1. \tag{17}$$

On the other hand, since $\rho_1^2 - 2x_1\rho_1 - 1 = 0$ and $\rho_2^2 - 2x_1\rho_2 + 1 = 0$, we see from Lemma 6 and (14) that

$$\left|\frac{t_1}{t_2}\log \rho_1 - \log \rho_2\right| > \exp\left(-5\cdot 10^8 D^4 \frac{S_1S_2}{D_1D_2} T^2 (\log E)^{-3}\right),$$

where $D_0 = 1$, $D = 2$, $D_2 = 2$, $D = 4$, $H_0 = t_2$, $H_1 = 2x_1$, $H_2 = 2x_1$, $B = t_2$, $A_1 = \rho_1$, $A_2 < \rho_1$, $S_0 = 1 + \log t_2$, $S_1 = 2 + \log \rho_1$, $s_2 < 2 + \log \rho_1$, $T < 5 + \log t_2 + \log(4(2 + \log \rho_1)^2)$, $E > e$. On combining this with (16) we obtain

(18)
$$\log 4 + 4^3\cdot 5\cdot 10^8(2+\log \rho_1)^2(5+\log t_2 + \log(4(2+\log \rho_1)^2))^2 > \log t_2 + 2t_2\log \rho_2.$$

Since $\log\log\log \rho_1 < \log\log \rho_1$, $\log(2 + \log \rho_1) < 2\log\log \rho_1$, $\rho_1 < \rho_2^{1\cdot 5}$, and $\log\log \rho_1 > \log\log x_1 > \log\log D^{1/4} > 1$ when $D > \exp 64$. Then from (18) we conclude

$$t_2 < e^{29}\log \rho_1(\log\log \rho_1)^2.$$

Substitute it into (17) we deduce

$$2e^{29}(\log\log \rho_1)^2 > \rho_1^2,$$

whence $\rho_1 < \exp 16\cdot 5$ and $D < x_1^4 < (\rho_1/2)^4 < \exp 64$. Thus the theorem. □

3. Proof of Theorem 2

Lemma 7 [6, Theorems 1 and 2]. *Let D' be a nonsquare integer, and let k be an integer with* g.c.d. $(k, D') = 1$. *If $D' > 0$, $|k| > 1$ and $2 \nmid k$, then every integer solution (X, Y, Z) of the equation*

$$X^2 - D'Y^2 = k^z, \qquad g.c.d.\ (X, Y) = 1, \qquad Z > 0 \tag{19}$$

can be expressed as

$$Z = Z_1 r, \qquad X + Y\sqrt{D'} = (X_1 \pm Y_1\sqrt{D^1})^r(\nu + w\sqrt{D'}),$$

where r *is a positive integer,* (ν, w) *is an integer solution of the equation*

$$\nu^2 - D'w^2 = 1, \tag{20}$$

(X_1, Y_1, Z_1) *is a positive integer solution of* (19) *which satisfies*

$$1 < \left|\frac{X_1 + Y_1\sqrt{D'}}{X_1 - Y_1\sqrt{D'}}\right| < (\nu_1 + w_1\sqrt{D'})^2$$

and

$$h(4D') \equiv 0 (\operatorname{mod} Z_1),$$

where $\nu_1 + w_1\sqrt{D'}$ *is the fundamental solution of* (20), $h(4D')$ *is the class number of binary quadric primitive forms with discriminant* $4D'$. □

Lemma 8 [2, Theorem 2]. *Let* $\alpha_1, \dots, \alpha_n$ *be algebraic numbers of the field* $\mathbb{K} = \mathbb{Q}(\alpha_1, \dots, \alpha_n)$ *with heights* $H_1, \dots, H_n$, *respectively,* $A_i = \max(4, H_i)$ $(i = 1, \dots, n)$, $A_1 \le \cdots \le A_{n-1} \le A_n$, *and let* d *denote the degree of* $\mathbb{K}$. *If* $\Lambda = b_1 \log\alpha_1 + \cdots + b_n \log\alpha_n \neq 0$ *for some integers* $b_1, \dots, b_n$, *then*

$$|\Lambda| > \exp\left(-(16nd)^{200n}(\log B)\left(\prod_{i=1}^{n}\log A_i\right)\left(\log\prod_{j=1}^{n-1}\log A_j\right)\right),$$

where $B = \max(4, |b_1|, \dots, |b_n|)$. □

Lemma 9 [1]. *Let* n *be an integer with* $n \ge 3$, *and let* $f(x) \in \mathbb{Z}[x]$ *with degree* m *and has at least two simple zeros. Then every integer solution* (x, y) *of*

$$f(x) = y^n$$

satisfies

$$\max(|x|, |y|) < \exp\exp((5n)^{10} m^{10m^3} H^{m^2}),$$

where H *is the height of* $f(x)$. □

Proof of Theorem 2. If m in (3) satisfies $2|m$, then

$$x^p = 2z^2 - 1, \tag{21}$$

where $z = (\varepsilon_1^{m/2} + \bar{\varepsilon}_1^{m/2})/2$ is an integer with $z > 1$. Then $(1, z, p)$ is an integer solution of the equation

$$X^2 - 2Y^2 = (-x)^z, \qquad \text{g.c.d. } (X, Y) = 1, \quad z > 0.$$

Since $3 + 2\sqrt{2}$ is the fundamental solution of the equation

$$\nu^2 - 2w^2 = 1 \tag{22}$$

and $h(8) = 1$. By Lemma 7, we have

$$1 + z\sqrt{2} = (X_1 \pm Y_1\sqrt{2})^p(\nu + w\sqrt{2}), \tag{23}$$

where (ν, w) is an integer solution of (22), X_1, Y_1 are positive integers and satisfy

$$X_1^2 - 2Y_1^2 = -x, \qquad \text{g.c.d.}\,(X_1, Y_1) = 1, \tag{24}$$

$$1 < \left|\frac{X_1 + Y_1\sqrt{2}}{X_1 - Y_1\sqrt{2}}\right| < (3 + 2\sqrt{2})^2. \tag{25}$$

Let $\varepsilon = X_1 + Y_1\sqrt{2}$, $\overline{\varepsilon} = X_1 - Y_1\sqrt{2}$, $\rho = 3 + 2\sqrt{2}$, $\overline{\rho} = 3 - 2\sqrt{2}$. From (23), (24), and (25), we obtain

$$1 + z\sqrt{2} = \begin{cases} \varepsilon^p \overline{\rho}^s, \\ -\overline{\varepsilon}^p \rho^s, \end{cases} \qquad 1 - z\sqrt{2} = \begin{cases} \overline{\varepsilon}^p \rho^s, \\ -\varepsilon^p \overline{\rho}^s, \end{cases} \quad ,$$

where s is an integer with $0 \le s \le p$. Hence

$$\varepsilon^p \overline{\rho}^s + \overline{\varepsilon}^p \rho^s = \pm 2$$

and

$$\left| p \log -\frac{\varepsilon}{\overline{\varepsilon}} - 2s \log \rho \right| = \frac{\sqrt{2}}{z} \sum_{l=0}^{\infty} \frac{(2y^2)^{-l}}{2l+1} < \frac{2\sqrt{2}}{z}. \tag{26}$$

Since $\rho^2 - 6\rho + 1 = 0$, $x(-\varepsilon/\overline{\varepsilon})^2 + 2(X_1^2 + 2Y_1^2)(-\varepsilon - |\overline{\varepsilon}) + x = 0$, and from (25) that $2(X_1^2 + 2Y_1^2) < 2\rho^2 x$. Hence the heights H_1, H_2 of ρ, $-\varepsilon/\overline{\varepsilon}$ satisfy $H_1 = 6$, $H_2 < 2\rho^2 x$. Since the degree d of $\mathbb{Q}(\rho, -\varepsilon - \overline{\varepsilon}) = \mathbb{Q}(\sqrt{2})$ is 2, by Lemma 8, we have

$$\begin{aligned} \left| p \log \frac{-\varepsilon}{\overline{\varepsilon}} - 2s \log \rho \right| &> \exp(-2^{2400}(\log 2p)(\log 2\rho^2 x)(\log 6)(\log\log 6)) \\ &> \exp(-2^{2400}(\log 2p)(\log 2\rho^2 x)). \end{aligned}$$

On combining this with (26) we get

$$\log 4 + 2^{2400}(\log 2p)(\log 2 + \log \rho + \log x) > \frac{1}{2}\log 2 + \log z > \frac{p}{2}\log x,$$

whence we conclude that

$$p < 2^{2415} \tag{27}$$

since $x > 1$. Further, by Lemma 9, from (21) we get

$$z = \max(x, z) < \exp\exp(2^{84}(5p)^{10}). \tag{28}$$

Since $z = (\varepsilon_1^{m/2} + \overline{\varepsilon}_1^{m/2})/2 > \sqrt{D}$, substitute (27) into (28) we get $D < \exp\exp \exp\exp 10$. The theorem is proved. □

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Research Department, Changsha Railway Institute, Changsha, Hunan, China