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MAXIMAL ABELIAN SUBALGEBRAS WITH SIMPLE NORMALIZER

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ABSTRACT. All infinite factors with separable predual contain a maximal Abelian * subalgebra whose normalizer generates a simple subfactor

1. INTRODUCTION

The purpose of this note is to point out that every infinite factor M, with separable predual, contains a maximal Abelian * subalgebra A whose normalizer $\mathscr{N}(A)$ generates a simple subfactor of M.

We recall that a subfactor $N \subset M$ is simple in M if $N \vee JNJ = B(L^2(M))$ where J is the modular conjugation of $L^2(M)$ (the lattice symbol $\vee$ denotes the von Neumann algebra generated). We refer to [2, 3] for the properties of simple subfactors; what we need to know here is that M always contains a simple injective subfactor.

The proof of our result closely follows an argument of Popa [4, p. 160] with one crucial difference: we use simple injective subfactors at the place of injective subfactors with trivial relative commutant [1].

In this way we obtain a superposition of the results in [2, 4] providing the general construction of a new kind of MASA whose properties are more stringent than those shared by semiregular MASA's [4]. (A is semiregular if $\mathscr{N}(A)''$ is a factor; this factor has automatically trivial relative commutant in M since it contains A. One calls A regular if $\mathscr{N}(A)'' = M$.)

2. THE CONSTRUCTION

Let $\mathscr{H}$ be a separable Hilbert space. We choose a bounded metric d on the unit ball $B(\mathscr{H})_1$ on $B(\mathscr{H})$, inducing the strong topology, and a strongly dense sequence $\{x_n\}$ in $B(\mathscr{H})_1$. If $N \subset B(\mathscr{H})$ is a von Neumann algebra we put

$$\delta(N) \equiv \sum_{i=1}^{\infty} \frac{d(x_i, N)}{2^i}$$

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where $d(\cdot, N)$ denotes the distance from the unit ball of N.

If N_k is an increasing sequence of von Neumann algebras then $\delta(N_k) \to 0$ iff $\vee N_k = B(\mathscr{H})$.

Let now M be an infinite factor on $\mathscr{H}$ with a cyclic separating vector Ω and modular conjugation J. If $A \subset M$ is an Abelian von Neumann subalgebra we put

$$\eta(A) = d([A\Omega], A \vee M'),$$

where $e = [A\Omega] \in A'$ denotes the projection onto the closure of $A\Omega$.

The following lemma is contained in [4, 5] and is included for convenience; the other lemmas are standard or elementary.

Lemma 1. *Let A_k be an increasing sequence of Abelian von Neumann subalgebras with $A = \vee A_k$.*

(i) *A is maximal Abelian in M iff $\eta(A) = 0$ i.e. $e \in A \vee M'$;*

(ii) *$\eta(A_k) \to \eta(A)$.*

Hence A is a MASA *of M iff $\eta(A_k) \to 0$.*

Proof.

(i) If A is a MASA of M, then $A' \cap M = A$ namely $A' = A \vee M'$ and $e \in A \vee M'$. Conversely assume $e \in A \vee M'$ and notice that the reduced von Neumann algebra A_e is maximal Abelian in $B(e\mathscr{H})$, i.e. $A'_e = A_e$, due to the cyclicity of Ω for A_e. Since $A' \cap M \supset A$ or $A' \supset A \vee M'$ we have

$$A_e = A'_e \supset (A \vee M')_e \supset A_e$$

thus $(A \vee M')_e = A_e$ or $(A' \cap M)_e = A_e$ that implies $A' \cap M = A$ because Ω is separating for M.

(ii) The sequence of projections $e_k = [A_k\Omega]$ converges to e increasingly, hence strongly, and $d(e_k, e) \to 0$; we have

$$\begin{aligned} \eta(A) - \eta(A_k) &= d(e, A \vee M') - d(e_k, A_k \vee M') \\ &= [d(e, A \vee M') - d(e, A_k \vee M')] + [d(e, A_k \vee M') - d(e_k, A_k \vee M')] \\ &\leq d(e, A \vee M') - d(e, A_k \vee M') + d(e, e_k) \to 0 \end{aligned}$$

where $d(e, A_k \vee M') \to d(e, A \vee M')$ because $A_k \vee M'$ increases to $A \vee M'$. □

Lemma 2. *Let N be an infinite factor with separable predual. There exists an increasing sequence of discrete Abelian von Neumann algebras B_n such that $B = \vee B_n$ is maximal Abelian in N and all the atoms of B_n are infinite (thus equivalent) projections of N.*

Proof. The statement is clear in the case of a type I factor F (consider the step function approximation of $L^\infty[0, 1]$ and regard it as a MASA of $B(L^2[0, 1])$ as usual). For a general infinite factor N note that for any MASA B of N the isomorphism of the diffuse part of B (if nonzero) with $L^\infty[0, 1]$ makes possible an atomic approximation that we only need to adjust in order that all projections be infinite. Since N is isomorphic to $N \otimes F$ we may achieve this by tensoring B with a MASA of F as above (the tensor product of two projections is an infinite projection if one of them is infinite). □

Lemma 3. *Let N be a factor and B a discrete Abelian von Neumann subalgebra of N. If all the atoms of B are equivalent, there exists a type I subfactor G of N such that B is a* MASA *of G. If the atoms of B are infinite projections, then $G' \cap N$ is an infinite factor.*

Proof. Let $\{p_i, i \in I\}$ be the atoms of B; fix $i_0 \in I$ and choose a partial isometry $v_i \in N$ from p_{i_0} to p_i, $i \in I$. Then $\{v_i v_j^* ; i, j \in I\}$ is a system of matrix units for G. As usual N is isomorphic to $N_{p_{i_0}} \otimes G$ hence $G' \cap N$ is isomorphic to $N_{p_{i_0}}$ which is an infinite factor iff p_{i_0} is an infinite projection. □

Lemma 4. *Let N_i be a subfactor of the factor M_i $(i = 1, 2)$. The subfactor $N_1 \otimes N_2$ of $M_1 \otimes M_2$ is simple iff N_1 is simple in M_1 and N_2 is simple in M_2.*

Proof. Let J_i be the modular conjugation of $L^2(M_i)$, so that $J = J_1 \otimes J_2$ is the modular conjugation of $L^2(M_1 \otimes M_2) = L^2(M_1) \otimes L^2(M_2)$. We have

$$(N_1 \otimes N_2) \vee J(N_1 \otimes N_2)J = (N_1 \vee J_1 N_1 J_1) \otimes (N_2 \vee J_2 N_2 J_2)$$

that readily entails the statement. □

Theorem 5. *Let M be an infinite factor with separable predual. There exists a maximal Abelian *subalgebra A of M whose normalizer $\mathcal{N}(A)$ generates a simple subfactor $\mathcal{N}(A)''$ of M.*

Proof. We order the pairs (A, F) consisting of a type I sufactor F of M with infinite relative commutant $F' \cap M$ and a maximal Abelian *subalgebra A of F in such a way that $(A, F) \subset (\tilde{A}, \tilde{F})$ means that $F \subset \tilde{F}$ and $\tilde{A} = A \vee B$ with B a MASA of $F' \cap \tilde{F}$ (in other words (A, F) is a tensor product component of $(\tilde{A}, \tilde{F})$). We will construct an increasing sequence of pairs (A_k, F_k) with

$$\eta(A_k) \to 0, \qquad \delta(F_k \vee J F_k J) \to 0$$

that will prove the theorem because $A = \vee A_k$ will be a MASA of M by Lemma 1 and $\mathcal{N}(A)'' \supset R$ where $R = \vee F_k$ is simple injective subfactor of M.

By an iterative argument it suffices to prove separately that, for any given pair (A, F), there exists a pair $(\tilde{A}, \tilde{F}) \supset (A, F)$ such that

(a) $\eta(\tilde{A}) \leq \frac{1}{2}\eta(A)$

(b) $\delta(\tilde{F} \vee J\tilde{F}J) \leq \frac{1}{2}\delta(F \vee JFJ)$.

To prove a) we choose in the factor $F' \cap M$ an increasing sequence of discrete Abelian von Neumann subalgebras B_n such that $\vee B_n$ is maximal Abelian and all the atoms of B_n are infinite, thus equivalent, in $F' \cap M$ (Lemma 2).

Since $A \vee B_n$ increases to a MASA of M we have $\eta(A \vee B_n) \to 0$. Let m be so large that $\eta(A \vee B_m) \leq \frac{1}{2}\eta(A)$ and let G be a type I subfactor of $F' \cap M$ containing B_m as a MASA and notice that the relative commutant of G in

$F' \cap M$ is infinite (Lemma 3). The pair $(\tilde{A}, \tilde{F})$ with $\tilde{A} = A \vee B_m$, $\tilde{F} = F \vee G$ satisfies a).

To prove b) let R be a simple injective subfactor of $F' \cap M$ [2]. Because of the tensor product decomposition $M \simeq F \otimes (F' \cap M)$ the subfactor $F \vee R$ of M is simple and injective (Lemma 4).

Let $\{F_n\}$ be an increasing sequence of type I subfactors of M, with $F = F_1$ and $F_n' \cap M$ infinite, that generates R. Since $\delta(F_n \vee JF_nJ) \to 0$ we may chose m so large that $\delta(F_m \vee JF_mJ) \leq \frac{1}{2}\delta(F \vee JFJ)$. Choose a MASA B of $F' \cap F_m$ and set $\tilde{A} = A \vee B$, $\tilde{F} = F_m$ so that $(\tilde{A}, \tilde{F}) \supset (A, F)$ and step b) is done. □

Remarks. Let A be MASA of M as in the theorem:

(a) If there exists a normal conditional expectation on ε of M onto A then A is a Cartan subalgebra of M. In fact if ϕ is a faithful normal state such that its modular group σ^ϕ leaves A invariant, then σ^ϕ leaves $\mathscr{N}(A)''$ invariant, therefore by Takesaki criterium there exists a normal conditional expectation of M onto $\mathscr{N}(A)''$ which implies $\mathscr{N}(A)'' = M$ [2].

(b) Since $\mathscr{N}(A)$ determines the automorphisms of M [2], it is possible to analyze the automorphism group of the pair as in [6]. For example denote by $\mathrm{Aut}(M|A)/\mathrm{Inn}(M|A)$ the group of automorphisms of M leaving A pointwise invariant modulo the corresponding inner automorphism subgroup; given $\alpha \in \mathrm{Aut}(M|A)$ the map $u \in \mathscr{N}(A) \to Z_u^\alpha \equiv \alpha(u)u^*$ is an ad-cocycle with values in A, that induces an isomorphism of $\mathrm{Aut}(M|A)/\mathrm{Inn}(M|A)$ into cohomology group $H^1_{\mathrm{ad}}(\mathscr{N}(A), A)$.

(c) The proof of the theorem shows that there exists a simple injective subfactor $R \subset M$ such that A is a regular MASA of R. If M is already approximately finite-dimensional one obtains (by a slight variation of the argument) the known result that M contains a regular MASA.

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