PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 107, Number 1, September 1989

MAXIMAL ABELIAN SUBALGEBRAS WITH SIMPLE NORMALIZER

ROBERTO LONGO

(Communicated by John B. Conway)

ABSTRACT. All infinite factors with separable predual contain a maximal Abelian * subalgebra whose normalizer generates a simple subfactor

1. INTRODUCTION

The purpose of this note is to point out that every infinite factor M, with separable predual, contains a maximal Abelian * subalgebra A whose normalizer $\mathscr{N}(A)$ generates a simple subfactor of M.

We recall that a subfactor $N \subset M$ is simple in M if $N \vee JNJ = B(L^2(M))$ where J is the modular conjugation of $L^2(M)$ (the lattice symbol $\vee$ denotes the von Neumann algebra generated). We refer to [2, 3] for the properties of simple subfactors; what we need to know here is that M always contains a simple injective subfactor.

The proof of our result closely follows an argument of Popa [4, p. 160] with one crucial difference: we use simple injective subfactors at the place of injective subfactors with trivial relative commutant [1].

In this way we obtain a superposition of the results in [2, 4] providing the general construction of a new kind of MASA whose properties are more stringent than those shared by semiregular MASA's [4]. (A is semiregular if $\mathscr{N}(A)''$ is a factor; this factor has automatically trivial relative commutant in M since it contains A. One calls A regular if $\mathscr{N}(A)'' = M$.)

2. THE CONSTRUCTION

Let $\mathscr{H}$ be a separable Hilbert space. We choose a bounded metric d on the unit ball $B(\mathscr{H})_1$ on $B(\mathscr{H})$, inducing the strong topology, and a strongly dense sequence $\{x_n\}$ in $B(\mathscr{H})_1$. If $N \subset B(\mathscr{H})$ is a von Neumann algebra we put

$$\delta(N) \equiv \sum_{i=1}^{\infty} \frac{d(x_i, N)}{2^i}$$

Received by the editors July 26, 1988 and, in revised form October 28, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L10; Secondary 46L35.
Key words and phrases. von Neumann algebras, subfactors, MASA.
Supported in part by Ministero della Pubblica Istruzione and CNR-GNAFA.

where $d(\cdot, N)$ denotes the distance from the unit ball of N.

If N_k is an increasing sequence of von Neumann algebras then $\delta(N_k) \to 0$ iff $\vee N_k = B(\mathscr{H})$.

Let now M be an infinite factor on $\mathscr{H}$ with a cyclic separating vector Ω and modular conjugation J. If $A \subset M$ is an Abelian von Neumann subalgebra we put

$$\eta(A) = d([A\Omega], A \vee M'),$$

where $e = [A\Omega] \in A'$ denotes the projection onto the closure of $A\Omega$.

The following lemma is contained in [4, 5] and is included for convenience; the other lemmas are standard or elementary.

Lemma 1. *Let A_k be an increasing sequence of Abelian von Neumann subalgebras with $A = \vee A_k$.*

(i) *A is maximal Abelian in M iff $\eta(A) = 0$ i.e. $e \in A \vee M'$;*

(ii) *$\eta(A_k) \to \eta(A)$.*

Hence A is a MASA *of M iff $\eta(A_k) \to 0$.*

Proof.

(i) If A is a MASA of M, then $A' \cap M = A$ namely $A' = A \vee M'$ and $e \in A \vee M'$. Conversely assume $e \in A \vee M'$ and notice that the reduced von Neumann algebra A_e is maximal Abelian in $B(e\mathscr{H})$, i.e. $A'_e = A_e$, due to the cyclicity of Ω for A_e. Since $A' \cap M \supset A$ or $A' \supset A \vee M'$ we have

$$A_e = A'_e \supset (A \vee M')_e \supset A_e$$

thus $(A \vee M')_e = A_e$ or $(A' \cap M)_e = A_e$ that implies $A' \cap M = A$ because Ω is separating for M.

(ii) The sequence of projections $e_k = [A_k\Omega]$ converges to e increasingly, hence strongly, and $d(e_k, e) \to 0$; we have

$$\begin{aligned}\eta(A) - \eta(A_k) &= d(e, A \vee M') - d(e_k, A_k \vee M') \\ &= [d(e, A \vee M') - d(e, A_k \vee M')] + [d(e, A_k \vee M') - d(e_k, A_k \vee M')] \\ &\le d(e, A \vee M') - d(e, A_k \vee M') + d(e, e_k) \to 0\end{aligned}$$

where $d(e, A_k \vee M') \to d(e, A \vee M')$ because $A_k \vee M'$ increases to $A \vee M'$. □

Lemma 2. *Let N be an infinite factor with separable predual. There exists an increasing sequence of discrete Abelian von Neumann algebras B_n such that $B = \vee B_n$ is maximal Abelian in N and all the atoms of B_n are infinite (thus equivalent) projections of N.*

Proof. The statement is clear in the case of a type I factor F (consider the step function approximation of $L^\infty[0, 1]$ and regard it as a MASA of $B(L^2[0, 1])$ as usual). For a general infinite factor N note that for any MASA B of N the isomorphism of the diffuse part of B (if nonzero) with $L^\infty[0, 1]$ makes possible an atomic approximation that we only need to adjust in order that all projections be infinite. Since N is isomorphic to $N \otimes F$ we may achieve this by tensoring B with a MASA of F as above (the tensor product of two projections is an infinite projection if one of them is infinite). □

Lemma 3. *Let N be a factor and B a discrete Abelian von Neumann subalgebra of N. If all the atoms of B are equivalent, there exists a type I subfactor G of N such that B is a* MASA *of G. If the atoms of B are infinite projections, then $G' \cap N$ is an infinite factor.*

Proof. Let $\{p_i, i \in I\}$ be the atoms of B; fix $i_0 \in I$ and choose a partial isometry $v_i \in N$ from p_{i_0} to p_i, $i \in I$. Then $\{v_i v_j^*; i, j \in I\}$ is a system of matrix units for G. As usual N is isomorphic to $N_{p_{i_0}} \otimes G$ hence $G' \cap N$ is isomorphic to $N_{p_{i_0}}$ which is an infinite factor iff p_{i_0} is an infinite projection. □

Lemma 4. *Let N_i be a subfactor of the factor M_i $(i = 1, 2)$. The subfactor $N_1 \otimes N_2$ of $M_1 \otimes M_2$ is simple iff N_1 is simple in M_1 and N_2 is simple in M_2.*

Proof. Let J_i be the modular conjugation of $L^2(M_i)$, so that $J = J_1 \otimes J_2$ is the modular conjugation of $L^2(M_1 \otimes M_2) = L^2(M_1) \otimes L^2(M_2)$. We have

$$(N_1 \otimes N_2) \vee J(N_1 \otimes N_2)J = (N_1 \vee J_1 N_1 J_1) \otimes (N_2 \vee J_2 N_2 J_2)$$

that readily entails the statement. □

Theorem 5. *Let M be an infinite factor with separable predual. There exists a maximal Abelian *subalgebra A of M whose normalizer $\mathcal{N}(A)$ generates a simple subfactor $\mathcal{N}(A)''$ of M.*

Proof. We order the pairs (A, F) consisting of a type I sufactor F of M with infinite relative commutant $F' \cap M$ and a maximal Abelian *subalgebra A of F in such a way that $(A, F) \subset (\tilde{A}, \tilde{F})$ means that $F \subset \tilde{F}$ and $\tilde{A} = A \vee B$ with B a MASA of $F' \cap \tilde{F}$ (in other words (A, F) is a tensor product component of $(\tilde{A}, \tilde{F})$). We will construct an increasing sequence of pairs (A_k, F_k) with

$$\eta(A_k) \to 0, \qquad \delta(F_k \vee J F_k J) \to 0$$

that will prove the theorem because $A = \vee A_k$ will be a MASA of M by Lemma 1 and $\mathcal{N}(A)'' \supset R$ where $R = \vee F_k$ is simple injective subfactor of M.

By an iterative argument it suffices to prove separately that, for any given pair (A, F), there exists a pair $(\tilde{A}, \tilde{F}) \supset (A, F)$ such that

(a) $\eta(\tilde{A}) \leq \frac{1}{2}\eta(A)$

(b) $\delta(\tilde{F} \vee J\tilde{F}J) \leq \frac{1}{2}\delta(F \vee JFJ)$.

To prove a) we choose in the factor $F' \cap M$ an increasing sequence of discrete Abelian von Neumann subalgebras B_n such that $\vee B_n$ is maximal Abelian and all the atoms of B_n are infinite, thus equivalent, in $F' \cap M$ (Lemma 2).

Since $A \vee B_n$ increases to a MASA of M we have $\eta(A \vee B_n) \to 0$. Let m be so large that $\eta(A \vee B_m) \leq \frac{1}{2}\eta(A)$ and let G be a type I subfactor of $F' \cap M$ containing B_m as a MASA and notice that the relative commutant of G in

$F' \cap M$ is infinite (Lemma 3). The pair $(\tilde{A}, \tilde{F})$ with $\tilde{A} = A \vee B_m$, $\tilde{F} = F \vee G$ satisfies a).

To prove b) let R be a simple injective subfactor of $F' \cap M$ [2]. Because of the tensor product decomposition $M \simeq F \otimes (F' \cap M)$ the subfactor $F \vee R$ of M is simple and injective (Lemma 4).

Let $\{F_n\}$ be an increasing sequence of type I subfactors of M, with $F = F_1$ and $F_n' \cap M$ infinite, that generates R. Since $\delta(F_n \vee JF_nJ) \to 0$ we may chose m so large that $\delta(F_m \vee JF_mJ) \leq \frac{1}{2}\delta(F \vee JFJ)$. Choose a MASA B of $F' \cap F_m$ and set $\tilde{A} = A \vee B$, $\tilde{F} = F_m$ so that $(\tilde{A}, \tilde{F}) \supset (A, F)$ and step b) is done. □

Remarks. Let A be MASA of M as in the theorem:

(a) If there exists a normal conditional expectation on ε of M onto A then A is a Cartan subalgebra of M. In fact if ϕ is a faithful normal state such that its modular group σ^ϕ leaves A invariant, then σ^ϕ leaves $\mathcal{N}(A)''$ invariant, therefore by Takesaki criterium there exists a normal conditional expectation of M onto $\mathcal{N}(A)''$ which implies $\mathcal{N}(A)'' = M$ [2].

(b) Since $\mathcal{N}(A)$ determines the automorphisms of M [2], it is possible to analyze the automorphism group of the pair as in [6]. For example denote by $\mathrm{Aut}(M|A)/\mathrm{Inn}(M|A)$ the group of automorphisms of M leaving A pointwise invariant modulo the corresponding inner automorphism subgroup; given $\alpha \in \mathrm{Aut}(M|A)$ the map $u \in \mathcal{N}(A) \to Z_u^\alpha \equiv \alpha(u)u^*$ is an *ad*-cocycle with values in A, that induces an isomorphism of $\mathrm{Aut}(M|A)/\mathrm{Inn}(M|A)$ into cohomology group $H^1_{\mathrm{ad}}(\mathcal{N}(A), A)$.

(c) The proof of the theorem shows that there exists a simple injective subfactor $R \subset M$ such that A is a regular MASA of R. If M is already approximately finite-dimensional one obtains (by a slight variation of the argument) the known result that M contains a regular MASA.

References

1. R. Longo, *Solution of the factorial Stone–Weierstrass conjecture. An application of the theory of standard split inclusions*, Invent. Math. **76** (1984), 145–155.

2. ——, *Simple injective subfactors*, Adv. in Math. **63** (1987), 152–171.

3. ——, *Restricting a compact action to an injective subfactor*, Ergodic Theory and Dynamical Systems (to appear).

4. S. Popa, *Semiregular maximal abelian subalgebras and the solution of the factor state Stone–Weierstrass problem*, Invent. Math. **76** (1984), 157–161.

5. C. Skau, *Finite subalgebras of a von Neumann algebra*, J. Funct. Anal. **25** (1977), 211–235.

6. I. M. Singer, *Automorphisms of finite factors*, Amer. J. Math. **77** (1955), 117–133.

Dipartimento di Matematica, Universitá di Roma II "Tor Vergata," Roma, Italy