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THE CAUCHY TRANSFORM ON BOUNDED DOMAINS

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ABSTRACT. Suppose that f is in $L^2(\Delta)$ where Δ is the unit disk, and that $f = 0$ outside Δ. We show that then the Cauchy transform $\mathscr{C}f$ of f, when restricted to Δ, satisfies $\|\mathscr{C}f\|_2 \le (2/\alpha)\|f\|_2$, where $\alpha \approx 2.4048$ is the smallest positive zero of the Bessel function J_0. This inequality is sharp.

1. INTRODUCTION

Let G denote a bounded domain in the complex plane $\mathbf{C}$ and let Δ be the unit disk $\{z: |z| < 1\}$. As usual, $L^2(G)$ denotes the space of complex-valued functions f, defined on G for which the norm

$$\|f\|_{2,G} = \left\{\int_G |f(x+iy)|^2\,dx\,dy\right\}^{1/2}$$

is finite. We denote $\|f\|_{2,\Delta}$ simply by $\|f\|_2$. Such a function f may be considered as an element of $L^2(\mathbf{C})$ by setting it equal to zero outside G. Then we may form the Cauchy transform

$$(\mathscr{C}f)(\zeta) = \frac{-1}{\pi}\int_G \frac{f(z)}{z-\zeta}\,dx\,dy$$

where $z = x + iy$. Unlike the two-dimensional Hilbert transform $\mathscr{H}f$ (also called the Beurling transform), the Cauchy transform is not bounded as an operator from $L^2(\mathbf{C})$ to $L^2(\mathbf{C})$. The characteristic function of any bounded domain has Cauchy transform whose modulus behaves like $|\zeta|^{-1}$ as $\zeta \to \infty$ and so does not belong to $L^2(\mathbf{C})$.

Nonetheless, it follows from the Sobolev embedding theorem (see, for example, [5, Theorem 2(ii), p. 124], or [4, p. 7]) that $\mathscr{C}$ is a bounded operator from $L^2(G)$ to $L^p(G)$ for all bounded plane domains G and all $p < \infty$. More precisely, we observe that if $g = \mathscr{C}f$ where $f \in L^2(G)$, then $\partial g/\partial \bar{z} = f \in L^2(\mathbf{C})$ while $\partial g/\partial z = \mathscr{H}f \in L^2(\mathbf{C})$ (see [3, (7.10), p. 157]). Now Sobolev's theorem

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implies that $g \in L^p(E)$ for all $p \in (0, \infty)$ and all compact subsets E of the plane. The fact that $\mathscr{C}f \in L^2(G)$ if $f \in L^2(G)$, has been used by Hamilton [2] to show the equivalence of the Poincaré inequality and the "analytic Poincaré inequality" for bounded domains G. (Hamilton formulates his result for bounded simply connected domains only, but the same proof works for all bounded domains.)

In this note we determine the precise value of the operator norm $\|\mathscr{C}\|$ when $\mathscr{C}$ is considered as an operator from $L^2(\Delta)$ into $L^2(\Delta)$. We denote by α the smallest positive zero of the Bessel function J_0 of order zero so that $\alpha \approx 2.4048256$ by [6, p. 748], and hence $2/\alpha \approx 0.83166$.

Theorem 1. *When $\mathscr{C}$ is considered as an operator from $L^2(\Delta)$ into $L^2(\Delta)$, we have $\|\mathscr{C}\| = 2/\alpha$.*

If the domain G is contained in a disk of radius R, then a linear change of variables shows that

$$\|\mathscr{C}f\|_{2,G} \leq (2R/\alpha)\|f\|_{2,G}$$

and that this inequality is sharp if G is a disk of radius R.

2. A problem on quadratic forms

To prove Theorem 1 it suffices to show that $\|\mathscr{C}P\|_2^2 \leq (2/\alpha)^2\|P\|_2^2$ whenever

$$P(z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn} z^m \bar{z}^n \tag{2.1}$$

is a polynomial in z and $\bar{z}$, since such functions are dense in $L^2(\Delta)$. In this case only finitely many of the complex numbers a_{mn} are nonzero. We set $P_{mn}(z) = z^m \bar{z}^n$, so that

$$(\mathscr{C}P_{mn})(\zeta) = \frac{-1}{\pi}\int_\Delta \frac{z^m \bar{z}^n}{z - \zeta}\,dx\,dy.$$

An application of Green's formula yields

$$(\mathscr{C}P_{mn})(\zeta) = \frac{-1}{2\pi i}\int_{\partial\Delta} \frac{z^m \bar{z}^{n+1}\,dz}{(n+1)(z-\zeta)} + \frac{1}{2\pi i}\lim_{\delta\to 0}\int_{\Gamma(\delta)} \frac{z^m \bar{z}^{n+1}\,dz}{(n+1)(z-\zeta)},$$

where $\partial\Delta$ is the unit circle and $\Gamma(\delta)$ denotes a circle of radius δ centered at ζ. We define

$$\varepsilon_p = \begin{cases} 1, & p \geq 1, \\ 0, & p \leq 0, \end{cases}$$

for integers p. Noting that $\bar{z}^{n+1} = z^{-n-1}$ for $|z| = 1$ and using the residue theorem we obtain

$$(\mathscr{C}P_{mn})(\zeta) = \frac{1}{n+1}[\zeta^m \bar{\zeta}^{n+1} - \varepsilon_{m-n}\zeta^{m-n-1}]. \tag{2.2}$$

If P is given by (2.1) then

$$
(2.3)\qquad
\begin{aligned}
\|P\|_2^2 &= \int_0^1 \int_0^{2\pi} \left| \sum_{m,n\geq 0} a_{mn} r^{m+n} e^{i(m-n)\theta} \right|^2 r\,dr\,d\theta \\
&= 2\pi \sum_{\substack{k,l,m,n\geq 0\\ m-n=k-l}} \frac{a_{mn}\overline{a_{kl}}}{m+n+k+l+2},
\end{aligned}
$$

whereas

(2.4)

$$
\begin{aligned}
\|\mathscr{C}P\|_2^2 &= \int_0^1 \int_0^{2\pi} \left| \sum_{m,n\geq 0} \frac{a_{mn}}{n+1} [z^m \bar{z}^{n+1} - \varepsilon_{m-n} z^{m-n-1}] \right|^2 r\,dr\,d\theta \\
&= 2\pi \sum_{\substack{k,l,m,n\geq 0\\ m-n=k-l}} \frac{a_{mn}\overline{a_{kl}}}{(n+1)(l+1)} \left\{ \frac{1}{m+n+k+l+4} - \frac{\varepsilon_{k-l}}{m+n+k-l+2} \right. \\
&\qquad \left. - \frac{\varepsilon_{m-n}}{m-n+k+l+2} + \frac{\varepsilon_{m-n}\varepsilon_{k-l}}{m-n+k-l} \right\}.
\end{aligned}
$$

For an integer p, let $\mathscr{S}_p$ denote the space of polynomials in z and $\bar{z}$ spanned by those functions $z^m \bar{z}^n$ for which $m - n = p$. Then if $f \in \mathscr{S}_i$, $g \in \mathscr{S}_j$ and $i \neq j$, it follows that f and g are orthogonal in the Hilbert space $L^2(\Delta)$. By (2.2), also $\mathscr{C}f$ and $\mathscr{C}g$ are then orthogonal, and we conclude that to find the norm of $\mathscr{C}$, it suffices to consider $\mathscr{C}f$ for $f \in \mathscr{S}_p$, for each integer p separately. This is evident also from (2.3) and (2.4).

So we set $m - n = k - l = p$ and for each p, we seek the best constant A_p such that, for all choices of $a_{mn} \in \mathbf{C}$ (with only finitely many a_{mn} nonzero), the inequality

(2.5)

$$
\begin{aligned}
&\sum_{n,l\geq \max(0,-p)} \frac{a_{mn}\overline{a_{kl}}}{(n+1)(l+1)} \left\{ \frac{1}{n+l+p+2} - \varepsilon_p \left[\frac{1}{n+p+1} + \frac{1}{l+p+1} - \frac{1}{p} \right] \right\} \\
&\qquad \leq A_p \sum_{n,l\geq \max(0,-p)} \frac{a_{mn}\overline{a_{kl}}}{n+l+p+1}
\end{aligned}
$$

holds. For convenience, we have multiplied the appropriate inequalities by 2 and used the fact that $\varepsilon_p^2 = \varepsilon_p$. Now set $\lambda_0 = (2/\alpha)^2 \approx 0.69166$. We show that, if $p \leq -1$ or $p \geq 2$ then (2.5) holds with $A_p = \frac{1}{2} < \lambda_0$ while if $p = 0$ or $p = 1$ then (2.5) holds with $A_p = \lambda_0$ and that λ_0 is the best possible constant in each of these two cases. This suffices for the proof of Theorem 1.

3. The cases $p \le -1$ and $p \ge 2$

Set $a_{n+p,n} = b_n$ and $a_{l+p,l} = b_l$ and consider first the case $p \le -1$ so that $\varepsilon_p = 0$. To prove (2.5) with $A_p = \frac{1}{2}$, we must show that $Q_1 \ge 0$, where

$$Q_1 = \sum_{n,l\ge -p} \frac{b_n\overline{b_l}}{(n+1)(l+1)} \left\{ \frac{(n+1)(l+1)}{2(n+l+p+1)} - \frac{1}{n+l+p+2} \right\}.$$

This can be rearranged to yield

$$Q_1 = \sum_{n,l\ge -p} \frac{b_n\overline{b_l}}{(n+1)(l+1)} \left\{ \frac{(n+1)(l+1)}{2(n+l+p+2)} - \frac{p+1}{(n+l+p+1)(n+l+p+2)} + \frac{(n-1)(l-1)}{2(n+l+p+1)(n+l+p+2)} \right\}$$

so that

$$Q_1 = \frac{1}{2}\int_0^1 \left| \sum_{n\ge -p} b_n r^n \right|^2 r^{p+1}\,dr - (p+1)\int_0^1 \left[\int_0^t \left| \sum_{n\ge -p} \frac{b_n}{n+1} r^n \right|^2 r^p\,dr \right] dt + \frac{1}{2}\int_0^1 \left[\int_0^t \left| \sum_{n\ge -p} \frac{(n-1)b_n}{n+1} r^n \right|^2 r^p\,dr \right] dt,$$

which is nonnegative for all choices of $\{b_n\}$ since $-(p+1) \ge 0$.

When $p \ge 2$ we have $\varepsilon_p = 1$ and we must show that $Q_2 \ge 0$ where

$$Q_2 = \sum_{n,l\ge 0} b_n\overline{b_l} \left\{ \frac{1}{2(n+l+p+1)} - \frac{1}{(n+1)(l+1)} \cdot \left[\frac{1}{n+l+p+2} - \frac{1}{n+p+1} - \frac{1}{l+p+1} + \frac{1}{p} \right] \right\}. \tag{3.1}$$

The expression in wave brackets can be rearranged to give

$$\frac{1}{2(n+l+p+1)} - \frac{n+l+2p+2}{p(n+p+1)(l+p+1)(n+l+p+2)}. \tag{3.2}$$

We set $N = n+p+1$, $L = l+p+1$ and $b_n = \gamma_N = \gamma_{n+p+1}$, $b_l = \gamma_L = \gamma_{l+p+1}$. Then N, $L \ge p+1$ and the inequality $Q_2 \ge 0$ becomes

$$\sum_{NL} \frac{\gamma_N\bar{\gamma}_L}{pNL} \left[\frac{pNL(N+L-p) - 2(N+L)(N+L-p-1)}{(N+L-p)(N+L-p-1)} \right] \ge 0.$$

Equivalently, we must show that $Q_3 \ge 0$ where

$$Q_3 = \sum_{NL} \frac{\delta_N\bar{\delta}_L}{p} \left[\frac{pNL - 2(N+L)}{N+L-p} + \frac{pNL}{(N+L-p)(N+L-p-1)} \right],$$

setting $\gamma_N/N = \delta_N$. Once again this can be rearranged to give

$$Q_3 = \sum_{NL} \delta_N\bar{\delta}_L \left\{ \frac{2(\tau N - \tau^{-1})(\tau L - \tau^{-1})}{p(N+L-p)} + \frac{(p^2-4)NL + 4p + 4(N-1)(L-1)}{p^2(N+L-p)(N+L-p-1)} \right\}$$

where $\tau = (p/2)^{1/2}$. Thus

$$Q_3 = \frac{2}{p}\int_0^1 \left|\sum_N (\tau N - \tau^{-1})\delta_N r^N\right|^2 r^{-p-1}\,dr$$
$$+ \frac{1}{p^2}\int_0^1 \left\{\int_0^t \left[4p\left|\sum_N \delta_N r^N\right|^2 + (p^2-4)\left|\sum_N N\delta_N r^N\right|^2\right.\right.$$
$$\left.\left. +4\left|\sum_N (N-1)\delta_N r^N\right|^2\right] r^{-p-2}\,dr\right\} dt$$

which is again nonnegative for all choices of δ_N since $p \geq 2$. Thus $Q_2 \geq 0$ for all choices of the parameters b_n as required.

4. An integral inequality for $p = 0$

When $p = 0$ we seek the smallest possible constant A_0 so that

$$\sum_{n,l\geq 0} \frac{b_n \overline{b_l}}{(n+1)(l+1)(n+l+2)} \leq A_0 \sum_{n,l\geq 0} \frac{b_n \overline{b_l}}{n+l+1}. \tag{4.1}$$

Set $\varphi(t) = \sum_n b_n t^n$ and $\psi(t) = \int_0^t \varphi(r)\,dr$ so that (4.1) becomes

$$\int_0^1 |\psi(r)|^2 \frac{dr}{r} \leq A_0 \int_0^1 |\varphi(r)|^2\,dr. \tag{4.2}$$

Since the quadratic forms in (4.1) are symmetric with real coefficients, it suffices to consider the inequality (4.2) for arbitrary real-valued continuous functions on $[0,1]$. So if $f \in C([0,1])$, we define $F(t) = \int_0^t f(r)\,dr$ and seek the best possible constant A_0 so that

$$\int_0^1 F(r)^2 \frac{dr}{r} \leq A_0 \int_0^1 f(r)^2\,dr. \tag{4.3}$$

This problem, which involves an inequality of Hardy type, can be solved by appealing to a more general result of Boyd ([1, Theorem 1, p. 368]). Consider the following eigenvalue problem (EP) in $(0,1)$:

(a) $\lambda y'' + y/x = 0$,
(b) $\lim_{x\to 0+} y(x) = 0$ and $\lim_{x\to 1-} y'(x) = 0$,
(c) $y \in C^2((0,1))$, $y(x) > 0$ and $y'(x) > 0$ in $(0,1)$.

Then the following auxiliary result holds.

Lemma 1. *There is a largest value of λ such that* (EP) *has a solution, and if λ^* denotes this value, then* (4.3) *holds for any $f \in L^2((0,1))$ with $A_0 = \lambda^*$. Equality holds if and only if f is a constant multiple of y'* (*almost everywhere*) *where y is the solution of* (EP) *corresponding to $\lambda = \lambda^*$.*

This is the special case $q = 0$, $r = p = 2$, $a = 0$, $b = 1$, $m(x) \equiv 1$, $w(x) = 1/x$ of [1, Theorem 1]. The additional compactness condition required is easily proved by using [1, inequality (14), p. 371]. Thus the best constant A_0 in (4.3) is the constant λ^* of Lemma 1.

The equation $\lambda y'' + y/x = 0$ has the solution $y(x) = x^{1/2} Z_1(2\sqrt{x/\lambda})$ by [6, formulas (5) and (6), p. 96], where Z_1 is a linear combination of the ordinary Bessel function J_1 and the Neumann function N_1. Condition (b) of (EP) excludes N_1 and so the solution is

$$y(x) = x^{1/2} J_1(2\sqrt{x/\lambda}).$$

Since $J_1(0) = 0$, all that remains is to choose the largest λ such that $y(x) > 0$ and $y'(x) > 0$ for $0 < x < 1$, with $y'(1) = 0$. Now with $\rho = 2/\sqrt{\lambda}$,

$$y'(x) = \frac{\rho}{2}\left\{J_1'(\rho\sqrt{x}) + \frac{J_1(\rho\sqrt{x})}{\rho\sqrt{x}}\right\}.$$

But for any z, we have $J_1'(z) + J_1(z)/z = J_0(z)$ by [6, formula (3), p. 45], with $\nu = 1$, so that

$$y'(x) = \frac{1}{2}\rho J_0(\rho\sqrt{x}) \quad \text{and} \quad \lambda^{1/2} y'(1) = J_0(2/\sqrt{\lambda}).$$

Thus the largest permissible value of λ is $\lambda^* = 4/\alpha^2$ where α is the smallest positive zero of J_0. We note that then also $y(x) > 0$ for $0 < x < 1$ since the smallest positive zero of J_1 is larger than that of J_0 ([6, section 15.22, p. 479]).

5. The case $p = 1$

When $p = 1$ we consider (3.1) and (3.2), replacing the appropriate factor $\frac{1}{2}$ by A_1. Again we may assume that the b_n are real, and we define $\varphi(r) = \sum_n b_n r^n$ and $\psi(r) = \sum_n b_n(1 - r^{n+1})/(n+1)$ so that $\psi(r) = \int_r^1 \varphi(t)\,dt$. A lengthy calculation shows that we are required to find the smallest constant A_1 such that

$$\int_0^1 \psi(r)^2\,dr \le A_1 \int_0^1 r\varphi(r)^2\,dr. \tag{5.1}$$

We set $t = 1 - r$ and $f(t) = \varphi(r) = \varphi(1-t)$. Then

$$F(t) \equiv \int_0^t f(u)\,du = \int_r^1 \varphi(v)\,dv = \psi(r).$$

Thus (5.1) is equivalent to

$$\int_0^1 F(t)^2\,dt \le A_1 \int_0^1 (1-t) f(t)^2\,dt \tag{5.2}$$

where f is real-valued and continuous.

To deal with (5.2), we apply Boyd's theorem and arrive at the differential equation

$$\lambda \frac{d}{dx}[y'(x)(1-x)] + y(x) = 0,$$

which needs to be satisfied together with (b) and (c) of (EP). Putting $1 - x = t$ we obtain

$$t\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{y}{\lambda} = 0. \tag{5.3}$$

From [6, formula (7) in section 4.31, p. 97], it follows that (5.3) has the solution $y(t) = J_0(2\sqrt{t/\lambda})$, so that $y(x) = J_0(2\sqrt{(1-x)/\lambda})$. Since we require that $y(0) = 0$, it is clear that we must choose $\lambda = 4/\beta^2$ where β is a positive zero of J_0. Hence the maximal λ is given by $\lambda = \lambda^* = 4/\alpha^2$. Since $J_0'(0) = 0$, we have $y'(1) = 0$. The condition that $y(x) > 0$ for $0 < x < 1$ also forces us to take $\beta = \alpha$ above. Finally, since $J_0'(u) = -J_1(u) < 0$ for $0 < u \leq \alpha$ (cf. [6, pp. 45, 479]), it follows that $y'(x) > 0$ for $0 < x < 1$, as required. We conclude that (5.1) and (5.2) hold with $A_1 = 4/\alpha^2$ and that this value of A_1 is best possible. Thus Theorem 1 is proved.

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