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A NOTE ON JOINT HYPONORMALITY

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ABSTRACT. We describe certain cones of polynomials in two variables naturally associated to the class(es) of operators T for which the tuple $(T, T^2, \dots, T^n)$ is jointly (weakly) hyponormal. As an application we give an example of an operator T such that the tuple (T, T^2) is jointly but not weakly hyponormal. Further, we show that there exists a polynomially hyponormal operator which is not subnormal if and only if there exists a weighted shift with the same property.

1. INTRODUCTION

Let H be a complex Hilbert space and $\mathscr{L}(H)$ be the bounded linear operators on H. An operator T is *hyponormal* provided that $[T^*, T] = T^*T - TT^*$ is positive semi-definite, or equivalently, provided that the operator matrix

$$\begin{bmatrix} I & T^* \\ T & T^*T \end{bmatrix}$$

is positive semi-definite. An n-tuple $\mathbf{T} = (T_1, \dots, T_n)$ of commuting operators where $T_i \in \mathscr{L}(H)$ is said to be *strongly hyponormal* if the block matrix

$$\begin{pmatrix} I & T_1^* & \cdots & T_n^* \\ T_1 & T_1^*T_1 & & T_n^*T_1 \\ T_2 & T_1^*T_2 & \cdots & \\ & & \ddots & \\ T_n & & & T_n^*T_n \end{pmatrix}$$

is positive semi-definite. An n-tuple $\mathbf{T} = (T_1, \dots, T_n)$ is *weakly hyponormal* if for each $\phi = (\phi_1, \dots, \phi_n) \in \mathbf{C}^n$ the operator $\sum_{i=1}^n \phi_i T_i$ is hyponormal. A single operator T is *strongly* (*weakly*) *n-hyponormal* provided that the n-tuple $(T, T^2, \dots, T^n)$ is strongly (weakly) hyponormal. These ideas have been considered by A. Ahtavale [At], J. Conway and W. Szymanski [CS], D. Xia [X], P. Szeptycksi [S] and others. Moreover, R. Curto, P. Muhly and J. Xia [CMX]

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have recently given an example of a pair $T = (T_1, T_2)$ of commuting operators which is weakly hyponormal but not strongly hyponormal.

In this note we establish correspondences between both cyclic strongly and weakly n-hyponormal operators and linear functionals which are positive on certain cones. This reduces questions about the existence or non-existence of cyclic weakly n-hyponormal operators which are not strongly n-hyponormal to problems about separating certain cones with linear functionals. Using these techniques, we give an example of a weighted shift operator which is weakly 2-hyponormal but not strongly 2-hyponormal. This result has also been obtained recently by Curto [C1].

An operator is said to be *polynomially hyponormal* or *weakly ∞-hyponormal*, if it is weakly n-hyponormal for all n. An operator $T \in \mathscr{L}(H)$ is *subnormal* if there exists a Hilbert space K containing H and $N \in \mathscr{L}(K)$ a normal operator for which $NH \subset H$ and $T = N|H$. Bram's theorem [B] says that T is subnormal if and only if T is strongly n-hyponormal, for all n. For notational consistency, we shall refer to subnormal operators as *strongly ∞-hyponormal.*

It is unknown whether every polynomial hyponormal operator is subnormal. However, we prove that there exists an operator which is polynomially hyponormal but not subnormal if and only if there exists a weighted shift with the same property.

A major component in the proof of these results is a construction of J. Agler [A1] which gives a one-one correspondence between pairs (T, γ) where $T \in \mathscr{L}(H)$ and $\gamma \in H$ and linear functionals acting on polynomials in two complex variables which obey certain positivity conditions. This construction and its content for strongly (weakly) n-hyponormal operators are presented in section two. In section three, we introduce "symmetrization" which allows us to reduce arguments to the case of weighted shifts. Section four is devoted to an elementary calculation which, combined with section three yields an example of a weighted shift which is weakly but not strongly 2-hyponormal.

2. Some cones of functions

Let $\mathbf{C}[z, w]$ denote the polynomials in two variables. We describe a construction of J. Alger (see [A1, Theorem 3.1]) which associates operators T on a Hilbert space H with linear functionals $\lambda\colon \mathbf{C}[z, w] \to \mathbf{C}$ which obey certain positivity conditions. Given $h(z, w) = \sum h_{ij} z^i w^j \in \mathbf{C}[z, w]$ and an operator $T \in \mathscr{L}(H)$, define $h(T, T^*) = \sum h_{ij} T^{*j} T^i$. If $x \in H$ we may define a linear functional $\Lambda_T\colon \mathbf{C}[z, w] \to \mathbf{C}$ by the formula $\Lambda_T(h) = \langle h(T, T^*)x, x\rangle$. Notice that $\Lambda(\overline{p(\overline{w})}p(z)) = \langle p(T)x, p(T)x\rangle \geq 0$. Moreover, if x is a cyclic vector for

T, then $\|T\| \leq 1$ if and only if

$$\begin{aligned} 0 &\leq \langle (I - T^*T)p(T)x, p(T)x\rangle \\ &= \langle p(T)^*(I - T^*T)p(T)x, x\rangle \\ &= \langle \overline{p(\overline{w})}(1 - zw)p(z)(T, T^*)x, x\rangle \\ &= \Lambda_T(\overline{p(\overline{w})}(1 - zw)p(z)). \end{aligned}$$

The following theorem is implicit in [A1, Theorem 3.3]. Roughly speaking, it says we may reverse the above computation.

Theorem 2.1 (Agler). *Suppose* $\Lambda\colon \mathbf{C}[z, w] \to \mathbf{C}$ *satisfies*

(1) $\Lambda(\overline{p(\overline{w})}p(z)) \geq 0$ *and*

(2) $\Lambda(\overline{p(\overline{w})}(1 - zw)p(z)) \geq 0$; *then there exists a Hilbert space* H, *an operator* $T \in \mathscr{L}(H)$ *which has a cyclic vector* $\gamma \in H$ *such that* $\|T\| \leq 1$ *and*

$$\Lambda(h) = \langle h(T, T^*)\gamma, \gamma\rangle \text{ for every } h \in \mathbf{C}[z, w].$$

Proof. As suggested above, one need only mimick the proof of [A1, Theorem 3.3]. □

If $T \in \mathscr{L}(H)$ has a cyclic vector γ, then T is weakly n-hyponormal if and only if for all polynomials p, q and ϕ, $\deg\phi < n + 1$,

$$\begin{aligned} 0 &\leq \left\langle \begin{bmatrix} I & \phi(T)^* \\ \phi(T) & \phi(T)^*\phi(T) \end{bmatrix} \begin{pmatrix} q(T)\gamma \\ p(T)\gamma \end{pmatrix}, \begin{pmatrix} q(T)\gamma \\ p(T)\gamma \end{pmatrix} \right\rangle \\ &= \langle (q(T)^*q(T) + q(T)^*\phi(T)^*p(T) + p(T)^*\phi(T)q(T) + \\ &\qquad p(T)^*\varphi(T)^*\varphi(T)p(T))\gamma, \gamma\rangle. \end{aligned}$$

In the notation of Theorem 2.1, T is weakly n-hyponormal if and only if

$$\begin{aligned} &\Lambda_T((\overline{q(\overline{w})} + \overline{p(\overline{w})}\phi(z))(q(z) + p(z)\overline{\phi(\overline{w})})) \\ &\quad = \langle (\overline{q(\overline{w})} + \overline{p(\overline{w})}\phi(z))(q(z) + p(z)\overline{\phi(\overline{w})})(T, T^*)\gamma, \gamma\rangle \geq 0. \end{aligned}$$

Define $\mathscr{E}_n \subseteq \mathbf{C}[z, w]$ for $i = 0, 1, 2, \ldots$ by

$$\begin{aligned} \mathscr{E}_0 &= \{\overline{p(\overline{w})}(1 - zw)p(z) | p \text{ is a polynomial}\} \\ \mathscr{E}_n &= \{(\overline{q(\overline{w})} + \overline{p(\overline{w})}\phi(z))(q(z) + p(z)\overline{\phi(w)}) | p, q \text{ are polynomials and} \\ &\quad \deg\phi < n + 1\}. \end{aligned}$$

Let $\mathscr{W}^n$ be the convex hull of the set $(\mathscr{E}_0 \cup \mathscr{E}_n)$. We have proven:

Proposition 2.2. *If* $T \in \mathscr{L}(H)$ *has cyclic vector* γ, *then* T *is a weakly* n-*hyponormal contraction if and only if* $\langle h(T, T^*)\gamma, \gamma\rangle \geq 0$ *for every* $h \in \mathscr{W}^n$.

For $n \geq 1$ let $\mathscr{E}^n = \{(\sum_{j=1}^k \overline{q_j(\overline{w})} z^j)(\sum_{i=1}^k q_i(z) w^i) | q_i$ are polynomials; $k < n + 1\}$. Denote by $\mathscr{S}^n$, the convex hull of $(\mathscr{E}_0 \cup \mathscr{E}^n)$.

Proposition 2.3. *If $T \in \mathcal{L}(H)$ has a cyclic vector γ, then T is a strongly n-hyponormal contraction if and only if $\langle h(T, T^*)\gamma, \gamma\rangle \geq 0$ for every $h \in \mathcal{S}^n$.*

Proof. Suppose T has a cyclic vector γ. Then T is strongly n-hyponormal if and only if for every n-tuple of polynomials $(q_1, q_2, \cdots, q_n)$

$$0 \leq \left\langle \begin{bmatrix} I & T^* & T^{*2} & \dots & T^{*n} \\ T & T^*T & T^{*2}T & \dots & T^{*n}T \\ T^2 & T^*T^2 & T^{*2}T^2 & & \vdots \\ & & & \ddots & \\ \vdots & & & & T^{*n}T^n \end{bmatrix} \begin{pmatrix} q_1(T)\gamma \\ q_2(T)\gamma \\ \vdots \\ q_n(T)\gamma \end{pmatrix}, \begin{pmatrix} q_1(T)\gamma \\ q_2(T)\gamma \\ \vdots \\ q_n(T)\gamma \end{pmatrix} \right\rangle$$

$$= \left\langle \left(\sum_j \overline{q_j(\overline{w})} z^j\right) \left(\sum_i q_i(z) w^i\right) (T, T^*)\gamma, \gamma \right\rangle .$$

Moreover, $\|T\| \leq 1$ if and only if $0 \leq \langle (\overline{p(\overline{w})}(1 - zw)p(z))(T, T^*)\gamma, \gamma\rangle$ for every polynomial p. □

Propositions 2.2 and 2.3 are no more than tautologies based upon the definitions. However, combined with the techniques of Theorem 2.1 they yield the following theorem which provides the basis for the example presented in Section 4.

Theorem 2.4. *The mapping $(T, \gamma) \to \Lambda_T$, establishes a one-to-one correspondence between the unitary equivalence classes of strongly (weakly) n-hyponormal contractions with fixed cyclic vector γ, and the linear functionals $\Lambda: \mathbf{C}[z, w] \to \mathbf{C}$ which are positive on $\mathcal{S}^n(\mathcal{W}^n)$.*

Proof. It is easily checked that $UT_1U^* = T_2$ and $U\gamma_1 = \gamma_2$ if and only if $\Lambda_{T_1} = \Lambda_{T_2}$. We need only show that every linear functional Λ which is positive on $\mathcal{S}^n$ gives rise to a strongly n-hyponormal contraction T with cyclic vector γ, such that $\Lambda = \Lambda_T$.

To this end, consider the space of polynomials in one variable $\mathbf{C}[z]$, and define a sesquilinear functional by setting

$$\langle p, q\rangle = \Lambda(\overline{q(\overline{w})}p(z)).$$

The fact that Λ is positive on $\mathcal{S}^n$ guarantees that $\langle , \rangle$ is positive semidefinite.

Let N be the null space of this sesquilinear form and let H be the Hilbert space obtained by completing $\mathbf{C}[z]/N$. It is straightforward to check that the properties of $\mathcal{S}^n$ guarantee that the operator of multiplication by z defines a contractive strongly n-hyponormal operator T with cyclic vector $\gamma = 1 + N$ and that $\Lambda_T = \Lambda$.

The proof for $\mathcal{W}^n$ is identical. □

Given two cones $\mathcal{C} \subseteq \mathcal{C}'$ in a vector space X we say that $\mathcal{C}$ and $\mathcal{C}'$ can be *separated*, if there exists a linear functional $\Lambda: X \to \mathbf{C}$ such that $\Lambda(h) \geq 0$ for every $h \in \mathcal{C}$, but $\Lambda(h') < 0$ for some $h' \in \mathcal{C}'$.

Corollary 2.5. *Let $1 \leq n \leq \infty$. There exists a cyclic weakly n-hyponormal operator which is not strongly n-hyponormal if and only if the cones $\mathscr{W}^n \subseteq \mathscr{S}^n$ in the vector space $\mathbf{C}[z, w]$ can be separated.*

We should remark that when $n = \infty$, the above corollary says that every cyclic polynomially hyponormal operator is subnormal if and only if the cones $\mathscr{W}^\infty \subseteq \mathscr{S}^\infty$ can not be separated. In this case, more can be said.

Corollary 2.6. *Every polynomially hyponormal operator is subnormal if and only if the cones $\mathscr{W}^\infty \subseteq \mathscr{S}^\infty$ can not be separated.*

Proof. One need only recall that an operator is subnormal if and only if its restriction to every cyclic invariant subspace is subnormal. □

3. Symmetrization

Define an operator L on polynomials in two variables by $L(\sum h_{ij} z^i w^j) = \sum h_{ii} z^i w^i$. L is known as the symmetrization operator (see Agler [A2]). L can be described analytically by

$$L(h) = \int_0^{2\pi} h(e^{i\theta} z, e^{-i\theta} w) \frac{d\theta}{2\pi}.$$

Given a symmetric polynomial in two variables, that is a polynomial of the form $\sum_{i=0}^n g_i z^i w^i$, define $P(\sum_{i=0}^n g_i z^i w^i) = \sum_{i=0}^n g_i x^i$. Let $R = P \circ L$. $R\colon \mathbf{C}[z, w] \to \mathbf{C}[x]$ is given explicitly by $R(\sum h_{ij} z^i w^j) = \sum h_{ii} x^i$, where $\mathbf{C}[x]$ denotes the collection of polynomials in the variable x.

Lemma 3.1. *Let $1 \leq n \leq \infty$. If $R(\mathscr{W}^n)$ can be separated from $R(\mathscr{S}^n)$, then there exists an operator which is weakly n-hyponormal but not strongly.*

Proof. Given $\lambda\colon \mathbf{C}[x] \to \mathbf{C}$ which separates $R(\mathscr{W}^n)$ from $R(\mathscr{S}^n)$ define $\Lambda\colon \mathbf{C}[z, w] \to \mathbf{C}$ by $\Lambda(h) = \lambda(R(h))$. Then Λ separates $\mathscr{W}^n$ from $\mathscr{S}^n$. □

In the case that $n = \infty$, the converse of Lemma 3.1 is true. For $p(x) = \sum p_i x^i \in \mathbf{C}[x]$, and an operator $T \in \mathscr{L}(H)$ let $p(T) = \sum p_i T^{*i} T^i$.

Proposition 3.2. *Every polynomially hyponormal operator is subnormal if and only if $R(\mathscr{W}^\infty)$ and $R(\mathscr{S}^\infty)$ can not be separated in $\mathbf{C}[x]$.*

Proof. If every polynomially hyponormal operator is subnormal, then by Corollary 2.6, $\mathscr{W}^\infty$ and $\mathscr{S}^\infty$ can not be separated, and hence $R(\mathscr{W}^\infty)$ and $R(\mathscr{S}^\infty)$ can not be separated.

Now, let T be a polynomially hyponormal contraction operator on the Hilbert space H which is not subnormal.

For $h \in \mathbf{C}[z, w]$ define $h_\theta = h_\theta(z, w) = h(e^{i\theta} z, e^{-i\theta} w)$. Note that if $h \in \mathscr{W}^\infty$, then $h_\theta \in \mathscr{W}^\infty$. Thus, for each $\gamma \in H$ and $h \in \mathscr{W}^\infty$, $\langle h_\theta(T, T^*)\gamma, \gamma\rangle \geq 0$. Whence,

$$0 \leq \int \langle h_\theta(T, T^*)\gamma, \gamma\rangle d\theta = \langle R(h)(T)\gamma, \gamma\rangle.$$

Further, since T is not subnormal, by a Theorem of Lambert [La], there exists a $\gamma \in H$ such that the weighted shift S with weights $\|T^{n+1}\gamma\|/\|T^n\gamma\|$ is not subnormal. Therefore, there exists $h \in \mathscr{S}^\infty$ such that $\langle h(S, S^*)e_0, e_0\rangle < 0$, where e_0 is the initial vector of S. Thus $0 > \langle h(S, S^*)e_0, e_0)\rangle = \int \langle h_\theta(T, T^*)\gamma, \gamma\rangle d\theta = \langle R(h)(T)\gamma, \gamma\rangle$. We have shown that the linear functional $\lambda(R(h)) = \langle R(h)(T)\gamma, \gamma\rangle$ separates $R(\mathscr{W}^\infty)$ from $R(\mathscr{S}^\infty)$. □

An operator T on a Hilbert space H is said to be a weighted shift if T has a cyclic vector $\gamma \in H$ for which $\langle T^n\gamma, T^m\gamma\rangle = 0$ if $n \neq m$. In this case, the vectors $\{T^k\gamma/\|T^k\gamma\|\}$ form an orthonormal sequence. The numbers w_k such that

$$T\left\{\frac{T^k\gamma}{\|T^k\gamma\|}\right\} = w_k\left\{\frac{T^{k+1}\gamma}{\|T^{k+1}\gamma\|}\right\}$$

are the weights of T.

Given an operator T with a cyclic vector γ define $\lambda_T\colon \mathbf{C}[x] \to \mathbf{C}$ by $\lambda_T(p) = \langle p(T)\gamma, \gamma\rangle = \sum\langle p_i T^{*i}T^i\gamma, \gamma\rangle$ for $p(x) = \sum p_i x^i$.

Proposition 3.3. *Let* $1 \leq n \leq \infty$. *The mapping* $(T, \gamma) \to \lambda_T$ *defines a one-to-one correspondence between unitary equivalence classes of contractive strongly (weakly) n-hyponormal weighted shifts with fixed cyclic vector* γ *and linear functionals* $\lambda\colon \mathbf{C}[x] \to \mathbf{C}$ *with* $\lambda(R(\mathscr{S}^n)) \geq 0 (\lambda(R(\mathscr{W}^n)) \geq 0)$.

Proof. It is easily checked that for weighted shifts $UT_1U^* = T_2$ and $U\gamma_1 = \gamma_2$ if and only if $\lambda_{T_1} = \lambda_{T_2}$.

Define $E\colon \mathbf{C}[x] \to \mathbf{C}[z, w]$ via $E(\sum p_i x^i) = \sum p_i z^i w^i$, so that for $p \in \mathbf{C}[x]$, $R \circ E(p) = p$, and for $h \in \mathbf{C}[z, w]$, $E \circ R(h) = L(h)$.

Given $\lambda\colon \mathbf{C}[x] \to \mathbf{C}$ with $\lambda(R(\mathscr{S}^n)) \geq 0$, define $\Lambda\colon \mathbf{C}[z, w] \to \mathbf{C}$ via $\Lambda(h) = \lambda(R(h))$, so that $\Lambda(\mathscr{S}^n) \geq 0$. Hence, by Theorem 2.4, there is a corresponding cyclic strongly n-hyponormal contraction T with cyclic vector γ. Since $\langle T^n\gamma, T^m\gamma\rangle = \Lambda(w^m z^n) = \lambda(R(w^m z^n)) = 0$, for $m \neq n$, we see that T is a weighted shift. Thus, the correspondence is onto.

The proof for the weak case is identical. □

Theorem 3.4. *Every polynomially hyponormal operator is subnormal if and only if every polynomially hyponormal weighted shift is subnormal.*

Proof. Apply Propositions 3.2 and 3.3. □

4. An example

In this section we apply the criteria developed in sections two and three to give an example of a weighted shift operator T which is weakly 2-hyponormal but not strongly two hyponormal. That is, the pair (T, T^2) is weakly two hyponormal but not strongly. Very recently Curto [C1] has also found an example of this. Further, Curto, Muhly, J. Xia [CMX] have recently given an example of a pair (T_1, T_2) which is weakly but not strongly hyponormal.

Theorem 4.1. *The shift operator* T *with weights*

$$w_n = \begin{cases} \sqrt{\frac{n+1}{n+2}} & \text{for } n \geq 1, \\ \frac{1}{\sqrt{2(\frac{8}{9}-\varepsilon)}} & \text{for } n = 0, \end{cases}$$

where $0 < \varepsilon < 1/100$, *is weakly* 2*-hyponormal but not strongly* 2*-hyponormal.*

By Proposition 3.3 and Lemma 3.1, Theorem 4.1 follows from the following lemma.

Lemma 4.2. *Let* $\lambda: \mathbf{C}[x] \to \mathbf{C}$ *be given by* $\lambda(f) = \int_0^1 f\,dx - (1/9+\varepsilon)f(0)$ *for any* $0 < \varepsilon < 1/100$; *then* $\lambda(R(\mathscr{W}^2)) \geq 0$ *but* $\lambda((1-4x+(10)/3x^2)^2) < 0$.

For $p, q, \phi \in \mathbf{C}[z]$, let

$$\Phi(p) = R(\overline{p(\overline{w})}(1-zw)p(z)),$$
$$\Psi(p, q, \phi) = R((\overline{q(\overline{w})} + \overline{p(\overline{w})}\phi(z))(q(z) + p(z)\overline{\phi(\overline{w})}).$$

It is clear that $R(\mathscr{W}^n)$ is generated, as a cone, by $\{\Phi(p) | p \in \mathbf{C}[z]\} \cup \{\Psi(p, q, \phi) | p, q \in \mathbf{C}[z] \text{ and } \phi = \sum_{i=1}^n \phi_i z^i\}$. Thus a linear functional $\lambda: \mathbf{C}[x] \to \mathbf{C}$ is non-negative on $R(\mathscr{W}^n)$ if and only if $\lambda(\Phi(p)) \geq 0$ and $\lambda(\Psi(p, q, \phi)) \geq 0$ over all $p, q, \phi \in \mathbf{C}[z]$, $\deg\phi < n+1$. Given $p = \sum p_i z^i$ it is easy to compute

$$\Phi(p) = \sum |p_i|^2 x^i.$$

Given $p = \sum p_i z^i$, $q = \sum q_i z^i$ and $\phi = \sum_{i=1}^n \phi_i z^i$ compute

$$\Psi(p, q, \phi) = \sum_{k\geq 0} x^k |\overline{q}_K + \sum_i \overline{p}_{i+k}\phi_i x^i|^2 + \sum_{k\geq 1} x^k |\sum_i \overline{p}_i \phi_{k+i} x^i|^2.$$

For $\lambda: \mathbf{C}[x] \to \mathbf{C}$ a linear functional, we let $\lambda(x^i) = \lambda_i$.

Proof of Lemma 4.2. Fix $0 < \varepsilon < 1/100$. We have

$$\lambda(x^n) = \begin{cases} \frac{1}{n+1} & \text{if } n > 0, \\ \frac{8}{9} - \varepsilon & \text{if } n = 0. \end{cases}$$

Clearly $\lambda(x^n) > \lambda(x^{n+1})$. Thus, since $\lambda(\Phi(p)) = \sum_{i\geq 0}(\lambda_i - \lambda_{i+1})|p_i|^2$, $\lambda(\Phi(p)) \geq 0$ for all $p \in \mathbf{C}[z]$.

To show that $\lambda(\Psi(p, q, \phi)) \geq 0$ for $p, q \in \mathbf{C}[z]$ and $\phi = \phi_1 z + \phi_2 z^2$ we assume this is not the case, i.e., that we can find $\hat{q}$, $\hat{p}$ and $\hat{\phi}$ as above with $\lambda(\Psi(\hat{p}, \hat{q}, \hat{\phi})) < 0$. Clearly we may assume that $\Psi(\hat{p}, \hat{q}, \hat{\phi})(0) = 1$; i.e., that $|\hat{q}_0|^2 = 1$. Since

(4.3)

$$\lambda(\Psi(p, q, \phi)) = \lambda\left(\sum_{k=1}^N x^k \left|\overline{q}_k + \sum_i \overline{p}_{k+i}\phi_i x^i\right|^2\right) + \lambda\left(\sum_{k=1}^2 x^k \left|\sum_i \overline{p}_i \phi_{k+i} x^i\right|^2\right) + \lambda\left(\left|\overline{q}_0 + \sum_{i=1}^2 \overline{p}_i \phi_i x^i\right|^2\right).$$

We have

$$\varepsilon > \left[\int_0^1 |\hat{q}_0^- + \hat{p}_1^- \hat{\phi}_1 x + \overline{\hat{p}}_2 \hat{\phi}_2 x^2|^2\, dx - \frac{1}{9}\right] + \int_0^1 x|\hat{p}_0^- \hat{\phi}_1 + \hat{p}_1^- \hat{\phi}_2 x|^2\, dx$$
$$+ \int_0^1 x^2 |\hat{p}_0^- \hat{\phi}_2|^2\, dx + \int_0^1 x|\hat{q}_1 + \hat{p}_2^- \hat{\phi}_1 x + \hat{p}_3^- \hat{\phi}_2 x^2|^2\, dx \geq 0,$$

where each summand is non-negative. Whence,

$$\varepsilon > \int_0^1 x|\hat{p}_0^- \hat{\phi}_1 + \hat{p}_1^- \hat{\phi}_2 x|^2\, dx,$$
$$\varepsilon > \int_0^1 x|\hat{q}_1^- + \hat{p}_2^- \hat{\phi}_1 x + \hat{p}_3^- \hat{\phi}_2 x^2|^2\, dx,$$
$$\varepsilon > \int_0^1 |\hat{q}_0^- + \hat{p}_1^- \hat{\phi}_1 x + \hat{p}_2^- \hat{\phi}_2 x^2|^2\, dx - \frac{1}{9} \geq 0.$$

It follows that

$$\sqrt{\varepsilon}(30) > |\hat{p}_2^- \hat{\phi}_1|,$$
$$6\sqrt{\varepsilon} > |\hat{p}_1^- \hat{\phi}_2|,$$
$$\varepsilon > \left|\frac{1}{\sqrt{5}}\hat{p}_2^- \hat{\phi}_2 + \frac{\sqrt{5}}{4}\hat{p}_1^- \hat{\phi}_1 + \frac{\sqrt{5}}{3}\hat{q}_0^-\right|^2,$$
$$\varepsilon > \left|\frac{1}{\sqrt{48}}\hat{p}_1^- \hat{\phi}_1 + \frac{\sqrt{48}}{12}\hat{q}_0^-\right|^2.$$

Now we can make the following estimates

$$|\hat{p}_2^- \hat{\phi}_1 \hat{p}_1 \hat{\phi}_\phi^-| \leq \sqrt{\varepsilon}(30) 6\sqrt{\varepsilon} = 180\varepsilon,$$

and

$$|\hat{p}_2^- \hat{\phi}_1 \hat{p}_1 \hat{\phi}_2| = |\hat{p}_2^- \hat{\phi}_2||\hat{p}_1 \hat{\phi}_1^-| \geq \left(\frac{10}{3} - \sqrt{\varepsilon}\left(\frac{5}{6}\sqrt{48} + \sqrt{5}\right)\right)(4 - \sqrt{48}\sqrt{\varepsilon}).$$

We must have

$$180\varepsilon > \left(\frac{10}{3} - \sqrt{\varepsilon}\left(\frac{5}{4}\sqrt{48} + \sqrt{5}\right)\right)(4 - \sqrt{48}\sqrt{\varepsilon}).$$

However, for $0 < \varepsilon < 1/100$ this inequality does not hold. We conclude that λ is non-negative on $R(\mathscr{W}^2)$. Finally, $\lambda((1 - 4x + (10)/3x^2)^2) = 8/9 - \varepsilon - 4 + (10/3)(2/3) + 16/3 - 20/3 + 100/45 = -\varepsilon < 0$. □

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