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POLYNOMIALS OF GENERATORS OF INTEGRATED SEMIGROUPS

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ABSTRACT. We give general sufficient conditions on p and A, for $p(A)$ to generate an exponentially bounded holomorphic k-times integrated semigroup, where p is a polynomial and A is a linear operator on a Banach space. Corollaries include the following.

(1) If iA generates a strongly continuous group and p is a polynomial of even degree with positive leading coefficient, then $-p(A)$ generates a strongly continuous holomorphic semigroup of angle $\frac{\pi}{2}$. (2) If $-A$ generates a strongly continuous holomorphic semigroup of angle θ and p is an nth degree polynomial with positive leading coefficient, with $n(\frac{\pi}{2} - \theta) < \frac{\pi}{2}$, then $-p(A)$ generates a strongly continuous holomorphic semigroup of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$. (3) If $(-A)$ generates an exponentially bounded holomorphic k-times integrated semigroup of angle θ, and p and θ are as in (2), then $-p(A)$ generates an exponentially bounded holomorphic $(k+1)$-times integrated semigroup of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$.

1. INTRODUCTION

In this paper, we consider linear operators, A, on a Banach space, whose resolvents $(w - A)^{-1}$, are $O(w^k)$ outside some set K, and polynomials, p, such that $p(K)$ is contained in a sector, and show that $p(A)$ generates a holomorphic $(k+2)$-times integrated semigroup. When $p(A)$ is densely defined, it generates a $(k+1)$-times integrated semigroup (Theorem 4). This is applied to polynomials of generators of holomorphic k-times integrated semigroups (Theorem 9) and polynomials of generators of strongly continuous groups (Theorem 11), as described in the abstract. Even for $k = 0$, that is, for polynomials of generators of strongly continuous semigroups, these results are new, except for some special cases, which we will describe below.

If k is a natural number, then the strongly continuous family of bounded operators $\{S(t)\}_{t\geq 0}$, is an *exponentially bounded k-times integrated semigroup generated by A* if $S(0) = 0$, and there exists real w such that (w, ∞) is

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contained in the resolvent set of A, with

$$(r-A)^{-1} = r^k \int_0^\infty e^{-rt} S(t)\,dt, \qquad \text{for } r > w$$

(see Arendt [1], [2], Hieber and Kellermann [8], Neubrander [10], Thieme [15]). For convenience, we will say *0-times integrated semigroup* to mean a strongly continuous semigroup. When A has nonempty resolvent set, A generates an exponentially bounded k-times integrated semigroup if and only if the abstract Cauchy problem $u'(t) = A(u(t))$, $u(0) = x$ has a unique exponentially bounded solution, for all x in $D(A^{k+1})$ (see Neubrander [10], or [5]).

It is well-known that the square of a generator of a strongly continuous group generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2}$. (This is the special case of Theorem 11, when $p(t) = t^2$. See Goldstein [7, Chapter 2, 8.7].) The bounded holomorphic semigroup analogue of Theorem 9, for the special case of $p(t) = t^n$, is in [3], where a different method of proof was used. (See Corollary 5 of this paper.) The special case of Theorem 9, when $p(t) = t^2$, appears in Goldstein [6]. In that paper, it is also shown that, if $-A$ generates a cosine function, then $-A^{2n}$ generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2}$, for $n = 1, 2, \ldots$.

All operators are linear, on a Banach space, X. We will write e^{tA} for the semigroup generated by A, $D(A)$ for the domain of A. Basic material on strongly continuous semigroups may be found in Goldstein [7], Pazy [14], or van Casteren [16]. When p is a polynomial, $p(A)$ is defined in the obvious way: if $p(t) = \sum_{k=0}^n a_k t^k$, then $p(A) \equiv \sum_{k=0}^n a_k A^k$, with $D(p(A)) = D(A^n)$. All polynomials are complex valued.

2. Main results

Definition 1. $S_\theta = \{re^{i\phi} | r \geq 0,\ |\phi| < \theta\}$.

Definition 2. A C_0 (strongly continuous) semigroup is a C_0 *holomorphic semigroup of angle* $\theta (0 < \theta \leq \frac{\pi}{2})$ if it extends to a semigroup holomorphic in the interior of S_θ, and continuous on $\overline{S}_\psi$, whenever $\psi < \theta$.

A C_0 holomorphic semigroup of angle θ is a *bounded holomorphic semigroup* (BHS) *of angle* θ if it is bounded on S_ψ, whenever $\psi < \theta$.

Remark. It is well-known (see any of the references for C_0 semigroups) that $-A$ generates a BHS of angle θ if and only if $D(A)$ is dense, the spectrum of A is contained in $\overline{S}_{\pi/2-\theta}$ and for all $\psi > (\frac{\pi}{2} - \theta)$, $\{\|w(w-A)^{-1}\| | w \notin \overline{S}_\psi\}$ is bounded.

Definition 3. Suppose $\frac{\pi}{2} \geq \theta > 0$. The k-times integrated semigroup $S(t)$ is an *exponentially bounded holomorphic k-times integrated semigroup of angle* θ if it extends to a family of operators $\{S(z)\}_{z \in S_\theta}$ satisfying

(1) The map $z \to S(z)$, from S_θ into $B(X)$, is holomorphic.

(2) $d^k/dz^k S(z)$ is a semigroup.

(3) For all $\psi < \theta$, $S(z)$ is strongly continuous on $\overline{S}_\psi$.

(4) For all $\psi < \theta$, there exist finite M_ψ, w_ψ such that $\|S(z)\| \leq M_\psi e^{w_\psi |z|}$, for all z in S_ψ.

Remark. Essentially the same idea, as appears in Definition 3, when $D(A)$ is dense, is in Okazawa [13], where semigroups of class (H_n) are defined.

Remark. In [4], we show that, if $D(A)$ is dense, then $-A$ generates an exponentially bounded holomorphic k-times integrated semigroup of angle $\theta > 0$ if and only if for all $\psi > (\frac{\pi}{2} - \theta)$, there exists real c_ψ such that the spectrum of A is contained in $c_\psi + S_\psi$, with $\{\|w^{1-k}(w-A)^{-1}\| | w \notin c_\psi + S_\psi\}$ bounded. When $D(A)$ is not dense, this condition is sufficient to guarantee that $-A$ generates an exponentially bounded holomorphic $(k+1)$-times integrated semigroup of angle θ.

Theorem 4. *Suppose K is a subset of the complex plane containing the spectrum of B, k is a nonnegative integer, $\frac{\pi}{2} > \theta > 0$, $\{\|w^{1-k}(w-A)^{-1}\| | w \notin K\}$ is bounded, and q is a polynomial such that $q(K)$ is contained in S_θ. Then*

(a) *$-q(A)$ generates an exponentially bounded holomorphic $(k+1)$-times integrated semigroup of angle $(\frac{\pi}{2} - \theta)$.*

(b) *If $D(q(A))$ is dense, then $-q(A)$ generates an exponentially bounded holomorphic k-times integrated semigroup of angle $(\frac{\pi}{2} - \theta)$.*

(c) *If $k = 0$, $D(q(A))$ is dense, and $q(0) = 0$, then $-q(A)$ generates a BHS of angle $(\frac{\pi}{2} - \theta)$.*

Proof. There exists finite M such that

$$\|(w-A)^{-1}\| \leq M|w|^{k-1}, \qquad \forall w \notin K.$$

Let $p(t) = q(t) - q(0)$, $V = S_\theta - q(0)$. To prove the theorem, it is sufficient to show that the spectrum of $p(A)$ is contained in V, with $\{\|z^{1-k}(z-p(A))^{-1}\| | z \notin V\}$ bounded (see Remarks after Definitions 2 and 3).

Suppose z is not in V. Let $\{w_j\}_{j=1}^N$ be the (not necessarily distinct) zeroes of $z - p(w)$, that is,

$$z - p(w) = \prod_{j=1}^N (w_j - w), \qquad \forall \text{complex } w.$$

We have

$$z - p(A) = \prod_{j=1}^N (w_j - A).$$

For any j, since $p(w_j) = z$ is not in V, w_j is not in K. Thus $(w_j - A)$ is invertible, and $\|(w_j - A)^{-1}\| \leq M|w_j|^{k-1}$. Thus, $z - p(A)$ is invertible, and we obtain the following upper bound for $(z-p(A))^{-1}$.

$$\begin{aligned}\|(z-p(A))^{-1}\| &\le \prod_{j=1}^{N}\|(w_j-A)^{-1}\| \\ &\le M^N\prod_{j=1}^{N}|w_j|^{k-1} \\ &= M^N|z|^{k-1}.\end{aligned}$$

Thus, $\{\|z^{1-k}(z-p(A))^{-1}\| | z\notin V\}$ is bounded, proving the theorem. □

As an immediate corollary, we get the results of [3].

Corollary 5. *Suppose $-A$ generates a BHS of angle θ, and $n(\frac{\pi}{2}-\theta)<\frac{\pi}{2}$. Then $-A^n$ generates a BHS of angle $\frac{\pi}{2}-n(\frac{\pi}{2}-\theta)$.*

Proof. Suppose $\frac{\pi}{2}>\psi>n(\frac{\pi}{2}-\theta)$. Then, since $-A$ generates a BHS of angle θ, the spectrum of A is contained in $S_{\psi/n}$, and $\{\|z(z-A)^{-1}\| | z\notin S_{\psi/n}\}$ is bounded.

Let $q(t)=t^n$. Since A generates a BHS, $D(q(A))$ is dense. Also $q(S_{\psi/n})$ is contained in S_ψ, so by Theorem 4(c), $-A^n=-q(A)$ generates a BHS of angle $\frac{\pi}{2}-\psi$, whenever $\frac{\pi}{2}>\psi>n(\frac{\pi}{2}-\theta)$. This implies that $-A^n$ generates a BHS of angle $\frac{\pi}{2}-n(\frac{\pi}{2}-\theta)$. □

In order to apply Theorem 4 to more general polynomials of other generators, we need some elementary lemmas.

Lemma 6. *Suppose E is a subset of the complex plane, and $\theta\ge 0$. Then*

$$\overline{\lim_{R\to\infty}}\ \sup\{|\arg(z)| \,|\, z\in E, |z|=R\}\le\theta$$

if and only if for all $\psi>\theta$, there exists real c_ψ such that E is contained in $c_\psi+S_\psi$.

Proof. Suppose the $\overline{\lim}$ inequality holds, and $\psi>\theta$. There exists finite M such that $|\arg(z)|<\psi$, when z is in E and $|z|\ge M$. Thus, E is contained in $S_\psi\cup\{z\in\mathbf{C} \,|\, |z|\le M\}$, which may be shown to be contained in $-M(1+\cot\psi)+S_\psi$.

Conversely, suppose that, for all $\psi>\theta$, there exists real c_ψ such that E is contained in $c_\psi+S_\psi$. For any $\psi\le\pi$, it is not hard to see that $\overline{\lim}_{R\to\infty}\sup\{|\arg(z)| \,|\, z\in c_\psi+S_\psi,\ |z|=R\}$ equals ψ. Thus

$$\overline{\lim_{R\to\infty}}\ \sup\{|\arg(z)| \,|\, z\in E,\ |z|=R\}\le\psi,$$

for all $\psi>\theta$, which concludes the proof. □

Lemma 7. *If $p(t)=t^n+q(t)$, where q is a polynomial of degree less than n, $\theta\ge 0$ and c is real, then, for all $\psi>n\theta$, there exists real c_ψ such that $p(c+S_\theta)$ is contained in $c_\psi+S_\psi$.*

Proof. $\lim_{|z|\to\infty} p(c+z)/z^n = 1$. Thus,

$$\begin{aligned}&\overline{\lim_{R\to\infty}} \sup\{|\arg(z)|\,|\,z \in p(c+S_\theta), |z| = R\} \\ &\quad = \overline{\lim_{R\to\infty}} \sup\{|\arg(p(c+z))|\,|\,z \in S_\theta, |z| = R\} \\ &\quad = \overline{\lim_{R\to\infty}} \sup\{|\arg(z^n)|\,|\,z \in S_\theta, |z| = R\} \\ &\quad = n\theta.\end{aligned}$$

Applying Lemma 6 now gives the result. □

Lemma 8. *Suppose K equals $-c+S_\theta \cup c-S_\theta$, where c and θ are nonnegative, and $p(t) = t^{2n} + q(t)$, where q is a polynomial of degree less than $2n$. Then, for all $\psi > 2n\theta$, there exists real c_ψ such that $p(K)$ is contained in $c_\psi + S_\psi$.*

Proof. Let $K^+ = -c + S_\theta$. Since $K = K^+ \cup -K^+$, it is sufficient, by Lemma 6, to show that $\overline{\lim}_{r\to\infty} \sup\{|\arg(p(z))|\,|\,z \in \pm K^+\} = 2n$; this follows exactly as in the proof of Lemma 7. □

Theorem 9. *Suppose $-A$ generates an exponentially bounded holomorphic k-times integrated semigroup of angle θ, $p(t) = t^n + q(t)$, where q is a polynomial of degree less than n, and $n(\frac{\pi}{2} - \theta) < \frac{\pi}{2}$. Then*

(a) *$-p(A)$ generates an exponentially bounded holomorphic $(k+1)$-times integrated semigroup of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$.*
(b) *If $D(p(A))$ is dense, then $-p(A)$ generates an exponentially bounded holomorphic k-times integrated semigroup of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$.*
(c) *If $k = 0$, then $-p(A)$ generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$.*

Proof. Suppose $\frac{\pi}{2} > \psi > n(\frac{\pi}{2} - \theta)$. Choose ϕ such that $\frac{\psi}{n} > \phi > \frac{\pi}{2} - \theta$. Since $-A$ generates an exponentially bounded holomorphic k-times integrated semigroup of angle θ, there exists real c such that the spectrum of A is contained in $c + S_\phi$, and $\{\|w^{1-k}(w-A)^{-1}\|\,|\,w \notin c + S_\phi\}$ is bounded.

By Lemma 7, there exists real c_ψ such that $p(c+S_\phi)$ is contained in $c_\psi + S_\psi$. By Theorem 4(a), $c_\psi I - p(A)$ generates an exponentially bounded holomorphic $(k+1)$-times integrated semigroup of angle $\frac{\pi}{2} - \psi$.

Thus, whenever $\frac{\pi}{2} > \psi > n(\frac{\pi}{2} - \theta)$, $-p(A)$ generates an exponentially bounded holomorphic $(k+1)$-times integrated semigroup of angle $\frac{\pi}{2} - \psi$. This implies (a).

The same argument, using Theorem 4(b), implies (b).

For (c), note that, since $-A$ generates a C_0 semigroup, $D(p(A))$ is dense. Thus the argument above, with Theorem 4(c), implies (c). □

Corollary 10. *Suppose p is a polynomial with positive leading coefficient.*

(a) *If $-A$ generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2}$, then $-p(A)$ generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2}$.*

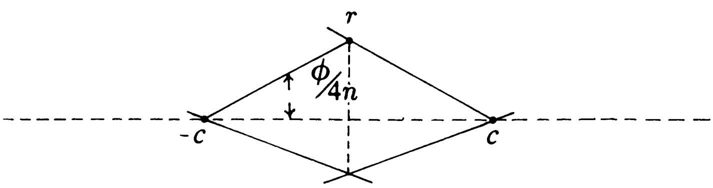
(b) *If* $-A$ *generates an exponentially bounded holomorphic* k*-times integrated semigroup of angle* $\frac{\pi}{2}$, *then* $-p(A)$ *generates an exponentially bounded holomorphic* $(k+1)$*-times integrated semigroup of angle* $\frac{\pi}{2}$.

The following theorem would follow from Theorem 9 and the fact that the square of the generator of a C_0 group generates a C_0 holomorphic semigroup, if $q(t)$ contained only even powers of t. Theorem 11 is more general, in that q may be any polynomial of degree less than $2n$.

Theorem 11. *Suppose* iA *generates a strongly continuous group, and* $p(t) = t^{2n} + q(t)$, *where* q *is a polynomial of degree less than* $2n$. *Then* $-p(A)$ *generates a* C_0 *holomorphic semigroup of angle* $\frac{\pi}{2}$.

Proof. Suppose $\frac{\pi}{2} > \phi > 0$. Since iA generates a C_0 group, there exists positive r such that the spectrum of A is contained in the horizontal strip $\{z \in \mathbf{C} \mid |\operatorname{Im}(z)| < r\}$, with $\{\| \operatorname{Im}(z)(z-A)^{-1}\| \mid |\operatorname{Im}(z)| \geq r\}$ bounded.

Let $c = r\cot(\frac{\phi}{4n})$, $K = (-c + S_{\phi/4n}) \cup (c - S_{\phi/4n})$.



Since $\{z \in \mathbf{C} \mid |\operatorname{Im}(z)| < r\}$ is contained in K, and $\{|z/\operatorname{Im}(z)| \mid z \notin K\}$ is bounded, it follows that $\{\|z(z-A)^{-1}\| \mid z \notin K\}$ is bounded.

By Lemma 8, there exists real c_ϕ such that $p(K)$ is contained in $c_\phi + S_\phi$.

Since iA generates a C_0 group, $D(p(A))$ is dense. By Theorem 4(c), $c_\phi I - p(A)$ generates a BHS of angle $\frac{\pi}{2} - \phi$. Thus, for any positive ϕ, $-p(A)$ generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2} - \phi$, so that $-p(A)$ generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2}$. □

The same argument, using Theorem 4(a) and (b), instead of (c), gives the following.

Theorem 12. *Suppose both* iA *and* $-iA$ *generate exponentially bounded* k*-times integrated semigroups and* p *is as in Theorem* 11. *Then*

(a) $-p(A)$ *generates an exponentially bounded holomorphic* $(k+1)$*-times integrated semigroup of angle* $\frac{\pi}{2}$.

(b) *If* $D(p(A))$ *is dense, then* $-p(A)$ *generates an exponentially bounded holomorphic* k*-times integrated semigroup of angle* $\frac{\pi}{2}$.

Corollary 13. *Suppose* p *is an arbitrary polynomial.*

(a) *If both* iA *and* $-iA$ *generate exponentially bounded* k*-times integrated semigroups, then* $-|p|^2(A)$ *generates an exponentially bounded holomorphic* $(k+1)$*-times integrated semigroup of angle* $\frac{\pi}{2}$.

(b) *If, in addition to* (*a*), $D(|p|^2(A))$ *is dense, then* $-|p|^2(A)$ *generates an exponentially bounded holomorphic* k*-times integrated semigroup of angle* $\frac{\pi}{2}$.

(c) *If* iA *generates a* C_0 *group, then* $-|p|^2(A)$ *generates a* C_0 *holomorphic semigroup of angle* $\frac{\pi}{2}$.

3. Examples

The most obvious application of Theorem 11 is to choose $A = i\frac{d}{dx}$, on $L^p(\mathbf{R})$, for $1 \le p < \infty$. For $p = \infty$, the resolvent of iA still satisfies the same growth conditions as the generator of a C_0 group, thus the proof of Theorem 11 with Theorem 4(a) replacing Theorem 4(c), gives us (b) of the following.

Example 1. Let $A \equiv i\frac{d}{dx}$, on $L^p(\mathbf{R})$ $(1 \le p \le \infty)$, with maximal domain, and $B \equiv (-1)^n(\frac{d}{dx})^{2n} + q(i\frac{d}{dx}) = A^{2n} + q(A)$, where q is a polynomial of degree less than $2n$.

(a) If $1 \le p < \infty$, then B generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2}$.

(b) If $p = \infty$, then B generates an exponentially bounded holomorphic once-integrated semigroup of angle $\frac{\pi}{2}$.

Remark. In Hieber and Kellermann [8], it is shown that $Q(i\frac{d}{dx})$, on $L^p(\mathbf{R})$, $1 \le p \le \infty$, generates an exponentially bounded once-integrated semigroup, whenever $Q(\mathbf{R})$ is contained in a left half-plane.

Suppose B generates an exponentially bounded k-times integrated semigroup. In Neubrander [11], and [5], it is shown that

$$(1) \qquad A \equiv \begin{bmatrix} B & B \\ 0 & B \end{bmatrix}, \qquad D(A) \equiv D(B) \times D(B),$$

generates an exponentially bounded $(k+1)$-times integrated semigroup. It is straightforward to show that this integrated semigroup is holomorphic if the integrated semigroup generated by B is (see Neubrander and deLaubenfels [12]). Thus we have the following, using the fact that

$$A^n = \begin{bmatrix} B^n & nB^n \\ 0 & B^n \end{bmatrix},$$

for all n.

Example 2. (a) Suppose both iB and $-iB$ generate exponentially bounded k-times integrated semigroups, and $p(t) = t^{2n} + q(t)$, where $\deg(q) < 2n$. Then

$$(2) \qquad -\begin{bmatrix} p(B) & Bp'(B) \\ 0 & p(B) \end{bmatrix}$$

generates a holomorphic exponentially bounded $(k+2)$-times integrated semigroup of angle $\frac{\pi}{2}$. If $D(B^{2n})$ is dense, then it generates a holomorphic exponentially bounded $(k+1)$-times integrated semigroup.

(b) Suppose $-B$ generates an exponentially bounded holomorphic k-times integrated semigroup of angle θ, and p is an nth degree polynomial with positive leading coefficient, with $n(\frac{\pi}{2} - \theta) < \frac{\pi}{2}$. Then the operator in (3.2) generates an exponentially bounded holomorphic $(k+2)$-times integrated semigroup of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$. If $D(B^n)$ is dense, then it generates an exponentially bounded holomorphic $(k+1)$-times integrated semigroup of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$.

In Example 2(a), B could be $Q(i\frac{d}{dx})$, on $L^p(\mathbf{R})(1 \le p \le \infty)$, when $Q(\mathbf{R})$ is contained in a left half-plane (see Remark after Example 1).

REFERENCES

1. W. Arendt, *Resolvent positive operators*, Proc. London Math. Soc. (3) **54** (1987), 321–349.
2. ——, *Vector valued Laplace transforms and Cauchy problems*, Israel J. Math. (3) **59** (1987), 327–352.
3. R. deLaubenfels, *Powers of generators of holomorphic semigroups*, Proc. Amer. Math. Soc. **99** (1987), 105–108.
4. ——, *Holomorphic C-semigroups and integrated semigroups*, J. d'Analyse Math. (submitted).
5. ——, *Integrated semigroups, C-semigroups and the abstract Cauchy problem*, Semigroup Forum (to appear).
6. J. A. Goldstein, *Some remarks on infinitesimal generators of analytic semigroups*, Proc. Amer. Math. Soc. **22** (1969), 91–93.
7. ——, *Semigroups of linear operators and applications*, Oxford, New York, 1985.
8. M. Hieber and H. Kellermann, *Integrated semigroups*, J. Func. An. (to appear).
9. I. Miyadera and N. Tanaka, *Some remarks on C-semigroups and integrated semigroups*, Ser. A, Proc. Japan Acad. **63** (1987).
10. F. Neubrander, *Integrated semigroups and their applications to the abstract Cauchy problem*, Pacific J. Math. (to appear).
11. ——, *Integrated semigroups and their applications to systems of evolution equations*, proceedings of a conference, in "Theory and Applications of Differential Equations," Marcel-Dekker (to appear).
12. F. Neubrander and R. deLaubenfels, *Laplace transform and regularity of C-semigroups*, preprint.
13. N. Okazawa, *Operator semigroups of class* (D_n), Math. Japonicae **18** (1973), 33–51.
14. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer, New York, 1983.
15. H. Thieme, *Integrated semigroups and duality*, preprint.
16. J. A. van Casteren, *Generators of strongly continuous semigroups*, in "Research Notes in Math.," Pitman, 1985.

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