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# THE ASYMPTOTICS OF THE DETERMINANT FUNCTION FOR A CLASS OF OPERATORS

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ABSTRACT. Let $A$ be an elliptic pseudodifferential operator on a closed manifold $M$ and $\operatorname{ord} A > \dim M$. We derive the asymptotics of $\log\det(1+\varepsilon A^{-1})$ when $\varepsilon \to \infty$. The constant term of this asymptotics equals $-\log\det A$.

## INTRODUCTION

Let $A$ be an elliptic pseudodifferential operator on a closed manifold $M$ and $n = \operatorname{ord} A > \dim M = d$. Suppose that $\arg(A\varphi, \varphi) \le \alpha$, $0 \le \alpha < \pi/2$. One can define complex powers of the $A$ and $A^{-s}$ are operators from the trace class when $\operatorname{Re} s > d/n$. The $\zeta$-function $\zeta(s) = \operatorname{Tr} A^{-s}$ is holomorphic in the half-plane $\operatorname{Re} s > d/n$ and admits analytic continuation to a meromorphic function in the whole complex plane. The poles of this function are located in the points $(d-j)/n$; $j = 0, 1, 2, \ldots$. The point $0$ is not really a pole of $\zeta(s)$. If the operator $A$ is differential or it is a power of a differential operator some other poles drop. The residues of $\zeta(s)$ and the numbers $\zeta(0)$, $\zeta(-1), \ldots$ can be calculated if one knows the complete symbol of the operator $A$ [1–3]. Particularly the point $s = 0$ is a regular point of the $\zeta$-function and one can define

$$W(A) = \log\det A = -\zeta'(0). \tag{1}$$

It is easy to check that this definition gives us the generalization of a finite dimensional operator's determinant.

On the other hand the inverse operator $A^{-1}$ belongs to the trace class and the determinants

$$D(\varepsilon) = \det(1+\varepsilon A^{-1}) = \prod_{j=1}^{\infty}(1+\varepsilon\lambda_j) \qquad (\lambda_j = \mu_j^{-1}) \tag{2}$$

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are defined. The infinite product in the right hand side of (2) converges. It is more convenient to consider the function

$$d(\varepsilon) = \log D(\varepsilon) = \sum_{j=1}^{\infty} \log(1 + \varepsilon\lambda_j). \tag{3}$$

We are going to investigate the asymptotic expansion of the $d(\varepsilon)$ when $\varepsilon \to +\infty$. In particular the determinant $W(A)$ will appear in this expansion.

We think there are two reasons why this expansion is interesting. The first reason is the connection between the Fredholm determinant $\det(1+\varepsilon A^{-1})$ and the determinant of the elliptic operator $A$ we shall obtain from this expansion. The second reason is that the value $\det(1 + \varepsilon A^{-1})$ appears in the measure theory as the result of integration of $\exp(-\varepsilon(x, x)/2)$ over a Hilbert space with respect to a Gaussian measure with average $0$ and correlation operator $A^{-1}$ (e.g. see [4]).

The fact that $\det(1 + \varepsilon A^{-1})$ admits asymptotic expansion for big $\varepsilon$ is not surprising. Let us differentiate formally the equality (3) with respect to $\varepsilon$:

$$d'(\varepsilon) = \sum_{j=1}^{\infty} \lambda_j (1 + \varepsilon\lambda_j)^{-1} = \sum_{j=1}^{\infty} (\mu_j + \varepsilon)^{-1} = \mathrm{Tr}(A + \varepsilon)^{-1}.$$

The trace of the resolvent $\mathrm{Tr}(A + \varepsilon)^{-1}$ admits asymptotic expansion when $\varepsilon \to \infty$ [5]. So one can justify the possibility of the last expansion's integration and receive the asymptotics for $d(\varepsilon)$. The only thing we can not get in this way is the constant of integration, i.e. the term with $\varepsilon^0$. We shall see later that this term is of the most interest. So we shall derive the asymptotics for $d(\varepsilon)$ directly.

The aim of this paper is to prove

**Theorem.** *The function $d(\varepsilon)$ admits asymptotic expansion*

$$d(\varepsilon) \sim \sum_{k=-d}^{\infty} p_k \varepsilon^{-k/n} + \sum_{j=0}^{\infty} q_j \varepsilon^{-j} \log \varepsilon \tag{4}$$

*with*

$$p_0 = -\log\det A. \tag{5}$$

## 2. Connection between $d(\varepsilon)$ and $\zeta(\varepsilon)$

In this section we shall prove

**Proposition 1.** *Let $d/n < a < 1$. Then*

$$d(\varepsilon) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, s=a} \varepsilon^s b(s)\zeta(s)\, ds \tag{6}$$

*where*

$$b(s) = \frac{1}{s} \int_0^{\infty} \frac{t^{-s}}{1+t}\, dt. \tag{7}$$

*Proof*. It is convenient to introduce the new argument $\omega = \log \varepsilon$. Let $d^*(\varepsilon) = d(e^\omega)$. Then formula (6) can be rewritten in the form

$$d^*(\omega) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, s=a} e^{s\omega} b(s)\zeta(s)\, ds. \tag{8}$$

Integrating by parts and changing the argument $t = e^\omega$ we derive

$$b(s) = \int_0^\infty t^{-s-1} \log(1+t)\, dt = \int_{-\infty}^\infty e^{-\omega s} \log(1+e^\omega)\, d\omega.$$

Note that the last formula gives the Fourier transformation $\log(1+e^\omega) \to b(is)$. So by the inverse Fourier formula

$$\log(1+e^\omega) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, s=a} b(s) e^{\omega s}\, ds.$$

After substituting $\varepsilon\lambda$ instead of $e^\omega$ we derive

$$\log(1+\varepsilon\lambda) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, s=a} \varepsilon^s b(s) \lambda^s\, ds.$$

Now to obtain (6) we have to sum the last identities for $\lambda = \lambda_j$. Note that the summation and integration operations in the right hand side commute and $\zeta(s)$ is the sum of $\lambda_j^s$.

## 3. Analytical properties of $b(s)$

We shall obtain asymptotic expansion for $d^*(\omega)$ (or $d(\omega)$) by shifting the contour in (8) to the left. So we must have analytical continuation of $b(s)$ into the left half-plane and we must estimate $|b(s)|$ and $|\zeta(s)|$ for large $|\mathrm{Im}\, s|$. In this section we shall prove

**Proposition 2.** *The function $b(s)$ admits analytical continuation to a meromorphic function into the half-plane $\mathrm{Re}\, s < 1$. It has a pole of order 2 into the point $s = 0$ and simple poles into the points $-1, -2, \ldots$; $\mathrm{Res}_{s=0}\, b(s) = 0$, $\mathrm{Res}_{s=0}\, sb(s) = 1$. The function $b(s)$ satisfies the following estimate*

$$|b(\sigma + i\tau)| \le C(\sigma)|\tau|^{\sigma-1} \exp\left(-\frac{\pi}{2}|\tau|\right), \qquad \sigma < 1, |\tau| \ge 1. \tag{9}$$

*Proof*. Let

$$J_n = \int_0^\infty t^{-s}(1+t)^n\, dt, \qquad n \ge 1.$$

Integrating by parts we obtain

$$J_n = -\frac{n}{s-1} \int_0^\infty \frac{t^{1-s}}{(1+t)^{n+1}}\, dt = -\frac{n}{s-1} J_n + \frac{n}{s-1} J_{n+1}.$$

So

$$\frac{s+n-1}{s-1} J_n = \frac{n}{s-1} J_{n+1}$$

and

$$J_n = \frac{n}{s+n-1} J_{n+1}.$$

Thus

$$J_1 = \frac{k!}{s(s+1)\cdots(s+k-1)} \int_0^\infty t^{-s}(1+t)^{-k-1}\,dt. \tag{10}$$

The integral in the right hand side of (10) is absolutely convergent in the strip $-k < \operatorname{Re} s < 1$. So the assertion about analytical continuation of $b(s)$ is proved.

Applying (10) with $k = 1$ we get

$$\operatorname{Res}_{s=0} sb(s) = \int_0^\infty \frac{dt}{(1+t)^2} = 1;$$

$$\operatorname{Res}_{s=0} b(s) = \frac{d}{ds}\int_0^\infty \frac{t^{-s}}{(1+t)^2}\,dt|_{s=0} = -\int_0^\infty \frac{\log t}{(1+t)^2}\,dt.$$

Changing the argument $t \to 1/u$ in the last integral one can see that

$$\int_0^\infty \frac{\log t}{(1+t)^2}\,dt = -\int_0^\infty \frac{\log u}{(1+u)^2}\,du,$$

so

$$\operatorname{Res}_{s=0} b(s) = 0.$$

To prove (9) we shall use the representation (10). The number $k$ in (10) will be large for large $|\tau|$ (really $k \sim |\tau|^2$). Suppose that $-\infty < \sigma_0 < \sigma < \sigma_1 < 1$ with some $\sigma_0$ and $\sigma_1$. All constants in the following estimates will depend on $\sigma_0$ and $\sigma_1$ only. We shall not numerate these constants. They will be designated by the same letter $C$.

To begin with, let us estimate the integral (10). This integral splits into

$$J = J_1 + J_2 = \int_0^{1/k} t^{-s}(1+t)^{-k-1}\,dt + \int_{1/k}^\infty t^{-s}(1+t)^{-k-1}\,dt.$$

The estimation of $J_1$ is very easy:

$$|J_1| \le C\int_0^{1/k} t^{-\sigma}\,dt \le Ck^{\sigma-1}.$$

To estimate $J_2$ let us suppose that $0 \le \sigma \le 1$ (note that we shift $s$ from the strip $\sigma_0 < \sigma < \sigma_1$!). The function $t^{-\sigma}$ decreases, so

$$|J_2| \le k^\sigma \int_{1/k}^\infty (1+t)^{-(k+1)}\,dt \le k^{\sigma-1}.$$

If $\sigma < 0$ we shall integrate by parts $n = |[\sigma]|$ times ( $[x]$—entire part of the number $x$ ).

$$\begin{aligned}|J_2| &\le \int_{1/k}^{\infty} t^{-\sigma}(1+t)^{-k-1}\,dt \\ &= k^{\sigma-1}\left(1+\frac{1}{k}\right)^{k} - \frac{\sigma}{k}\int_{1/k}^{\infty} t^{-\sigma-1}(1+t)^{-k}\,dt = \cdots \\ &= k^{\sigma-1}\left(1+\frac{1}{k}\right)^{-k} - \sigma\frac{k^{\sigma}}{k-1}\left(1+\frac{1}{k}\right)^{-(k-1)} + \cdots \\ &\quad + (-1)^n\frac{\sigma(\sigma+1)\cdots(\sigma+n-1)}{k(k-1)\cdots(k-n+1)}\int_{1/k}^{\infty} t^{-fr(\sigma)}(1+t)^{-k+n-1}\,dt.\end{aligned}$$

Every term except the last is obviously estimated by $Ck^{\sigma-1}$. The integral in the last term is the integral of the form we have just investigated ($k \mapsto k-n$; $\sigma \mapsto fr(\sigma)$ ), so it is estimated by $k^{fr(\sigma)-1}(fr(\sigma) = \sigma - [\sigma])$. The product which stands before the integral is of the order $k^{-n} = k^{-[\sigma]}$. Thus the last term is also estimated by $k^{\sigma-1}$. Finally,

$$|J_2| \le Ck^{\sigma-1}$$

and

$$\left|\int_0^{\infty} t^{-s}(1+t)^{-k-1}\,dt\right| \le Ck^{\sigma-1}. \tag{11}$$

Now we intend to estimate the product which stands before the integral in the right hand side of (10). Denote

$$r(k) = \log\left|\frac{k!}{s^2(s+1)\cdots(s+k-1)}\right|$$

and

$$l = |[\sigma_0]| + 1.$$

We have

$$\begin{aligned} r(k) &= \log k! - 2\log|s| - \sum_{j=1}^{l-1}\log|s+j| - \sum_{j=l}^{k-1}\log|s+j| \\ &\le \log k! - (l+1)\log|\tau| - \frac{1}{2}\sum_{j=1}^{k-1}\log((\sigma+j)^2+\tau^2) \\ &= \log k! - (l+1)\log|\tau| - \sum_{j=1}^{k-1}\log(\sigma+j) - \frac{1}{2}\sum_{j=l}^{k-1}\log\left(1+\frac{\tau^2}{(\sigma+j)^2}\right) \\ &= \log l! + \sum_{j=0}^{k-l-1}\log\left(\frac{k-j}{k-j+\sigma-1}\right) - \frac{1}{2}\sum_{j=l}^{k-1}\log\left(1+\frac{\tau^2}{(\sigma+j)^2}\right) \\ &\quad - (l+1)\log|\tau|. \end{aligned} \tag{12}$$

The first term in the right hand side of (12) is constant. The second term is bounded by $C + (1-\sigma)\log k$. Indeed,

$$\begin{aligned}
\sum_{j=0}^{k-1} \log\left(\frac{k-j}{k-j+\sigma-1}\right) &= \sum_{j=0}^{k-l-1} \log\left(1 + \frac{1-\sigma}{k-j+\sigma-1}\right) \\
&= \sum_{j=0}^{k-l-1} \log\left(1 + \frac{1-\sigma}{l+\sigma+j}\right) \\
&\le \log\left(1 + \frac{1-\sigma}{l+\sigma}\right) + \int_1^{k-l} \log\left(1 + \frac{1-\sigma}{x}\right) dx \\
&= C + x\log\left(1 + \frac{1-\sigma}{x}\right)\Bigg|_1^{k-l} \\
&\quad + (1-\sigma)\int_1^{k-l} \frac{dx}{(x+1-\sigma)} \\
&\le C + (1-\sigma)\log k\,.
\end{aligned}$$

To estimate the third term in the right hand side of (12) we note that the function $\log(1+\tau^2/(\sigma+j)^2)$ is decreasing with respect to $j$. Thus

$$\begin{aligned}
\frac{1}{2}\sum_{j=l}^{k-1} \log\left(1 + \frac{\tau^2}{(\sigma+j)^2}\right) &\ge \frac{1}{2}\int_l^k \log\left(1 + \frac{\tau^2}{(\sigma+x)^2}\right) dx \\
&= \frac{1}{2}\int_{l+\sigma}^{k+\sigma} \log\left(1 + \frac{\tau^2}{x^2}\right) dx \\
&= \frac{1}{2}(k+\sigma)\log\left(1 + \frac{\tau^2}{(\sigma+k)^2}\right) \\
&\quad - \frac{1}{2}(l+\sigma)\log\left(1 + \frac{\tau^2}{(\sigma+l)^2}\right) \\
&\quad + \tau\arctan\frac{k+\sigma}{\tau} - \tau\arctan\frac{l+\sigma}{\tau} \\
&\ge C + \frac{1}{2}(k+\sigma)\log\left(1 + \frac{\tau^2}{(\sigma+k)^2}\right) - (l+\sigma)\log|\tau| \\
&\quad + |\tau|\arctan\frac{k+\sigma}{|\tau|}\,.
\end{aligned}$$

Finally

$$
\begin{aligned}
r(k) &\le C + (1+\sigma)\log k - \frac{1}{2}(k+\sigma)\log\left(1 + \frac{\tau^2}{(\sigma+k)^2}\right) \\
&\quad + (l+\sigma)\log|\tau| - |\tau|\arctan\frac{k+\sigma}{|\tau|} - (l+1)\log|\tau| \\
&= C + (1-\sigma)\log k + (\sigma-1)\log|\tau| \\
&\quad - \frac{1}{2}(k+\sigma)\log\left(1+\frac{\tau^2}{(\sigma+k)^2}\right) - |\tau|\arctan\frac{k+\sigma}{|\tau|}.
\end{aligned} \tag{13}
$$

Now we take $k \sim \tau^2$. Then the second term in the right hand side of (13) equals $2(1-\sigma)\log|\tau|$ up to the additive constant. The fourth term is bounded. The fifth term equals

$$|\tau|\arctan(|\tau| + o(1)) = \frac{\pi}{2}|\tau| + o(1).$$

So

$$r(k) \le -\frac{\pi}{2}|\tau| + (1-\sigma)\log|\tau| + C\,; k \sim \tau^2. \tag{14}$$

Therefore after substitution $[\tau^2]$ instead of $k$ into (11) we obtain

$$|b(s)| \le C|\tau|^{2(\sigma-1)}\exp\left(-\frac{\pi}{2}|\tau| + (1-\sigma)\log|\tau|\right) = C|\tau|^{\sigma-1}\exp\left(-\frac{\pi}{2}|\tau|\right).$$

## 4. Estimation of $|\zeta(\sigma+i\tau)|$ for large $|\tau|$

**Proposition 3.** *If operator $A$ satisfies the assumptions of this paper then*

$$|\zeta(\sigma+i\tau)| \le C|\tau|^{-\sigma-1/2}\exp\left(\frac{\pi}{2}|\tau|\right), \qquad C = C(\sigma), |\tau| \ge 1.$$

*Proof.* We shall use the representation

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\theta(t)\,dt = \frac{J(s)}{\Gamma(s)} \tag{15}$$

with

$$\theta(t) = \mathrm{Tr}\, e^{-tA} = \sum_{j=1}^{\infty} e^{-t\mu_j}. \tag{16}$$

It is well known (e.g. see [2]) that

$$\theta(t) \sim e^{-at}, \qquad t \to \infty \tag{17}$$

and (Minakshisundaram-Plejel expansion)

$$\theta(t) \sim \sum_{j=-d}^{0} a_j t^{j/n} + \sum_{j=1}^{\infty}(a_j t^{j/n} + b_j t^{j/n}\log t), \qquad t \to 0. \tag{18}$$

Moreover one can differentiate both expansions (17) and (18). Split the integral from the right hand side of (15):

$$J(s) = J_1(s) + J_2(s) = \int_0^1 t^{s-1}\theta(t)\,dt + \int_1^\infty t^{s-1}\theta(t)\,dt.$$

To estimate $J_2$ we use partial integration

$$|J_2(s)| = \left|\frac{\theta(1)}{s} + \frac{1}{s}\int_1^\infty t^s\theta'(t)\,dt\right| \le \frac{1}{|s|}\left[\theta(1) + C\int_1^\infty t^\sigma e^{-at}\,dt\right]$$
$$\le \frac{C}{|\tau|}; \qquad C = C(\sigma).$$

To estimate $J_1$ we take $l = 1 + |[\sigma]|$. Then

$$\theta(t) = \sum_{j=-d}^{l}(a_j t^{j/n} + b_j t^{j/n}\log t) + r(t)$$

($b_j = 0$ if $j \le 0$) with

$$|r(t)| \le Ct^{(l+1)/n}|\log t| \quad \text{and} \quad |r'(t)| \le Ct^{l/n}|\log t|; \qquad 0 \le t \le 1 .$$

So

$$J_1(s) = \sum_{j=-d}^{l} a_j\int_0^1 t^{j/n+s-1}\,dt + \sum_{j=1}^{l} b_j\int_0^1 t^{j/n+s-1}\log t\,dt + \int_0^1 t^{s-1}r(t)\,dt$$
$$= \sum_{j=-d}^{l}\frac{a_j}{s+j/n} - \sum_{j=1}^{l}\frac{b_j}{(s+j/n)^2} + \frac{1}{s}r(1) - \frac{1}{s}\int_0^1 t^s r'(t)\,dt$$

and

$$|J_1(s)| \le C/|s| \le C/|\tau|.$$

Finally

$$|J(s)| \le \frac{C}{|\tau|}, \qquad C = C(\sigma). \tag{19}$$

To estimate $1/\Gamma(s)$ we use the Stirling asymptotics

$$\log\Gamma(s) = (z - 1/2)\log z - z + (1/2)\log(2\pi) + o(1); \qquad |z| \to \infty, |\arg z| < \pi$$

(e.g. see [6]). We can write

$$\log\Gamma(\sigma + i\tau) = [(\sigma - 1/2) + i\tau][(1/2)\log(\sigma^2 + \tau^2) + i\arg(\sigma + i\tau)]$$
$$- \sigma - i\tau + (1/2)\log(2\pi) + o(1)$$

and

$$|\Gamma(\sigma + i\tau)| \sim \sqrt{2\pi}e^{-\sigma}\exp((1/2)(\sigma - 1/2)\log(\sigma^2 + \tau^2) - \tau\arg(\sigma + i\tau)); \quad |\tau| \to \infty.$$

Note that

$$\arg(\sigma + i\tau) = \frac{\pi}{2}\operatorname{sgn}\tau - \frac{\sigma}{\tau} + o\left(\frac{1}{|\tau|}\right)$$

and

$$\log(\sigma^2+\tau^2) = 2\log|\tau| + o(1/|\tau|^2).$$

Therefore

(20) $$|\Gamma(\sigma+i\tau)| \sim \sqrt{2\pi}|\tau|^{\sigma-1/2}e^{-\pi|\tau|/2}; \qquad |\tau|\to\infty.$$

Now the assertion of Proposition 3 follows from (15), (19) and (20).

## 5. Proof of the theorem

Propositions 2 and 3 show us that the function which is integrated in (8) is bounded by $C|\tau|^{-3/2}$. So we can shift the path of integration to the left in the complex plane as far as we want. The poles of the function $b(s)\zeta(s)$ give us terms in asymptotics of $d^*(\omega)$. Denote

(21)
$$\begin{aligned}
\beta_j &= \operatorname{Res} b(s)|_{s=-j}, \qquad j=0,1,\dots; \qquad \hat\beta_0 = \operatorname{Res} sb(s)|_{s=0};\\
\hat\beta_j &= (b(s)-\beta_j/(s+j))|_{s=-j}, \quad j=1,2,\dots;\\
\alpha_k &= \operatorname{Res}\zeta(s)|_{s=-k/n}, \qquad k=-d,\,-d+1,\dots;\\
\gamma_j &= (\zeta(s)-\alpha_{nj}/(s+j))|_{s=-j}, \qquad j=1,2,\dots.
\end{aligned}$$

Note that $\alpha_0 = 0$ and if $A$ is a differential operator or it is a power of a differential operator then

$$\alpha_{nj}=0 \quad\text{and}\quad \gamma_j=\zeta(-j).$$

By Proposition 2

$$\beta_0 = 0 \quad\text{and}\quad \hat\beta_0 = 1.$$

The points $s=-k/n$, $k\neq jn(j=0,1,\dots)$ are simple poles of the function

$$F(s)=e^{s\omega}b(s)\zeta(s)$$

and

$$\operatorname{Res}_{s=-k/n}F(s) = \alpha_k e^{-k\omega/n}b(-k/n).$$

The points $s=jn$ are poles of the second order of $F(s)$. It is easy to calculate residues in these points:

$$\begin{aligned}
\operatorname{Res}_{s=0}F(s) &= \zeta'(0)+\omega\zeta(0),\\
\operatorname{Res}_{s=-j}F(s) &= \alpha_{nj}\beta_j\omega e^{-j\omega} + (\alpha_{nj}\hat\beta_j+\gamma_j\beta_j)e^{-j\omega}.
\end{aligned}$$

Thus we have obtained

(22) $$d^*(\omega)\sim\sum_{k=-d}^{\infty}p_k e^{-k\omega/n}+\sum_{j=0}^{\infty}q_j\omega e^{-j\omega}, \qquad \omega\to\infty$$

with

(23)
$$\begin{aligned}
p_k &= \alpha_k b(-k/n), k\neq jn(j=0,1,\dots);\\
p_0 &= \zeta'(0)\\
p_{jn} &= \alpha_{nj}\hat\beta_j+\gamma_j\beta_j, \qquad j=1,2,\dots;\\
q_0 &= \zeta(0); q_j=\alpha_{nj}\beta_j.
\end{aligned}$$

After substituting $e^{-\omega}=\varepsilon$ into (22) one obtains (4). In particular (5) holds.

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