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AFFINE INVARIANT SUBSPACES OF $C(\mathbf{C})$

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ABSTRACT. A linear subspace A of $C(\mathbf{C})$ is affine invariant if $f(z) \in A$ implies that $f(az+b) \in A$ for every a, $b \in \mathbf{C}$.

We present a classification of the affine invariant closed subspaces of $C(\mathbf{C})$, and of those affine invariant subspaces which are also composition invariant (i.e., $f, g \in A$ implies that $f \circ g \in A$).

1. INTRODUCTION AND PRELIMINARIES

Let K be a Hausdorff space and let G be a group which acts on K. Let $C(K)$ denote the linear topological space (over the complex field $\mathbf{C}$) of continuous $\mathbf{C}$ valued functions on K, with the topology of uniform convergence on compact subsets of K. A closed linear subspace A of $C(K)$ is said to be G-invariant if $f = f(x) \in A$ and $\tau \in G$ implies that $f \circ \tau = f(\tau x) \in A$.

In [5] Schwartz classifies the closed translation invariant subspaces of $C(\mathbf{R})$. His results do not apply to translation invariant subspaces of $\mathbf{R}^n$, $n \geq 2$. (See [G] and also [B-S-T] for related results.)

In this note we study the structure of closed linear subspaces A of $C(\mathbf{C})$ which are invariant under the group of affine transformations; i.e. if $f(z) \in A$ then $f(az+b) \in A$ for every a, b in $\mathbf{C}$; and present a complete classification of these spaces.

Our proofs are elementary and do not depend on Schwartz's theorem on mean periodic functons [S].

Definitions. 1. Let $\mathbf{N} = \{0, 1, 2, \dots\}$ denote the nonnegative integers and let $\mathbf{N}^* = \mathbf{N} \cup \{\infty\}$. We regard $\mathbf{N}^*$ as an ordered set by letting $n < \infty$ for $n \in \mathbf{N}$.

2. Set $M = \mathbf{N} \times \mathbf{N}$ and $M^* = \mathbf{N}^* \times \mathbf{N}^*$.

3. Let $\leq$ denote the partial order on M and M^* defined by $(n, k) \leq (n', k')$ if and only if $n \leq n'$ and $k \leq k'$.

4. A subset J of M is an order ideal if $(n, k) \in J$ and $(x, y) \leq (n, k)$ implies that $(x, y) \in J$.

5. Let J be a subset of M. $u \in M^*$ is an upper bound for J if $x \in J$, $u \leq x$ implies $u = x$. An upper bound u of J is a least upper bound if for

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any other upper bound v of J in M^*, $v \le u$ implies $v = u$. $\sup J$ is the set of all least upper bounds of J.

6. Let J be a subset of M. Define A_J to be the closed linear span in $C(\mathbf{C})$ of the monomials

$$\{z^n \overline{z}^k : (n, k) \in J\}.$$

7. Let $f \in C(\mathbf{C})$. The spectrum $\sigma(f)$ of f is defined by

$$\sigma(f) = \{(n, k) : z^n \overline{z}^k \in V_f\}$$

where V_f denotes the closed affine invariant subspace generated by f.

8. If A is a subset of $C(\mathbf{C})$ then $\sigma(A) = \bigcup\{\sigma(f) : f \in A\}$.

The following proposition is simple and is presented without a proof.

Proposition 1. (i) *The union and the intersection of any family of order ideals is an order ideal.*

(ii) *For* $(n,k) \in M^*$ $[(n,k)] = \{(x,y) \in M : (x,y) \le (n,k)\}$ *is an order ideal in* M *which will be called the basic ideal determined by* (n,k).

(iii) *If* $J \subset M$ *is an order ideal then the set* $\sup J \subset M^*$ *is finite. If* $\sup J = \{(n_j, k_j)\}_{j=1}^r$ *is ordered so that* $0 \le n_1 < n_2 < \cdots < n_{r-1} < n_r \le \infty$ *then* $\infty \ge k_1 > k_2 > \cdots > k_{r-1} > k_r \ge 0$, *and* $J = \bigcup_{j=1}^r [(n_j, k_j)]$. *So,* J *is the union of the maximal basic ideals that it contains.*

Our main result is the following theorem:

Theorem 1. *The correspondence* $J \to A_J$ *is a one-to-one mapping of the set of order ideals of* M *onto the set of closed affine invariant subspaces of* $C(\mathbf{C})$. *Its inverse is the correspondence* $A \to \sigma(A)$. *This correspondence is Boolean in the sense that* $J_1 \cap J_2 \to A_{J_1} \cap A_{J_2}$ *and* $J_1 \cup J_2 \to \overline{\text{span}}\{A_{J_1} \cup A_{J_2}\}$. *Consequently, each affine invariant closed linear subspace* A *of* $C(\mathbf{C})$ *admits a representation as* $A = \overline{\text{span}} \bigcup_{j=1}^r A_{[(n_j, k_j)]}$ *where* $\{(n_j, k_j)\}_{j=1}^r = \sup \sigma(A)$.

The following is a simple corollary. (A subset A of $C(\mathbf{C})$ is *composition invariant* if $f \circ g \in A$ whenever $f, g \in A$).

Corollary 1. *The only closed linear subspaces of* $C(\mathbf{C})$ *which are both affine invariant and composition invariant are the following five spaces.* $A_{[(0,0)]}$ = *the constants,* $A_{[(1,0)]}$ = *the affine functions,* $A_{[(1,1)]}$ = *the linear functions* (*i.e. functions of the form* $f(z) = az + b\overline{z} + c$), $A_{[(\infty,0)]}$ = *the entire analytic functions and* $C(\mathbf{C}) = A_{[(\infty,\infty)]}$.

Another corollary is the following theorem due to De Leeuw and Katznelson [D.L-K].

Corollary 2. *Let* K *be a compact Hausdorff space and let* B *be a closed linear subspace of* $C(K)$ *which contains the constants. Let* $h \in C(\mathbf{C})$ *operate on* B (*i.e.* $f \in B$ *implies* $h \circ f \in B$). *If* h *is nonlinear then* B *is an algebra while if* h *is nonanalytic then* B *is self-adjoint.*

Theorem 1 and the Corollaries will be proved in the next section.

2. Proofs

Let $A \subset C(\mathbf{C})$ be a closed affine invariant linear subspace. Then $A \cap C^{\infty}(\mathbf{C})$ is dense in A. Indeed, if $\{K_r\}_{r>0} \subset C^{\infty}(\mathbf{C})$ is a compactly supported positive summability kernel then the convolution $K_r * f$ is in $A \cap C^{\infty}(\mathbf{C})$ whenever $f \in A$ and converges to f in A when r tends to ∞.

If $f \in A \cap C^{\infty}(\mathbf{C})$ then $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z = \frac{1}{2}(\partial f/\partial x - i\partial f/\partial y)$ and $\partial f/\partial \overline{z} = \frac{1}{2}(\partial f/\partial x + i\partial f/\partial y)$ are all in A. ($\partial f/\partial x$ and $\partial f/\partial y$ are obtained by assuming $z = x + iy$. Note that for the above we only apply the translation invariance of A.) Also, $f(tz+a)$ is in A for all t, $a \in \mathbf{C}$ and thus $d/dt f(tz+a)$ is also in A, where d/dt stands for differentiation with respect to the real variable t.

The following is a characterization of $\sigma(f)$ for $f \in C^{\infty}(\mathbf{C})$. If $(n,k) \in M$ and $f \in C^{\infty}(\mathbf{C})$ we set $D^{n,k} f = \partial^{n+k}/\partial z^n \partial \overline{z}^k f$.

Proposition 2. *If $f \in C^{\infty}(\mathbf{C})$ and $(n,k) \in M$ then $(n,k) \in \sigma(f)$ if and only if $D^{n,k} f \not\equiv 0$.*

Proof. We observe that if $D^{n,k} f \equiv 0$ then each function g in V_f satisfies $D^{n,k} g \equiv 0$ implying that $z^n \overline{z}^k \notin V_f$. This proves the "only if" part. Conversely, assume that $D^{n,k} f \not\equiv 0$. We have

$$\frac{d^{n+k}}{dt^{n+k}} f(tz+a) = \sum_{j=0}^{n+k} \binom{n+k}{j} z^j \overline{z}^{n+k-j} D^{j,n+k-j} f(tz+a)$$

and for all $t \in \mathbf{R}$, $a \in \mathbf{C}$ this is an element of V_f. Set $t = 0$ and pick some $a \in \mathbf{C}$ so that $D^{n,k} f(a) \neq 0$. Then the polynomial

$$g(z) = \sum_{j=0}^{n+k} \binom{n+k}{j} (D^{j,n+k-j} f(a)) z^j \overline{z}^{n+k-j} = \sum_{j=0}^{n+k} c_j z^j \overline{z}^{n+k-j}$$

is in V_f and the coefficient $c_n = \binom{n+k}{n} D^{n,k} f(a)$ of $z^n \overline{z}^k$ in g is not 0. For any polynomial $h(z) = \sum_{j=1}^{m} c_j z^{n_j} \overline{z}^{k_j}$ in V_f and any $b \in \mathbf{C}$ $h(bz) - b^n \overline{b}^k h(z)$ is in V_f and satisfies $h(bz) - b^n \overline{b}^k h(z) = \sum_{j=1}^{m} c_j (b^{n_j} \overline{b}^{k_j} - b^n \overline{b}^k) z^{n_j} \overline{z}^{k_j}$. Thus, given any $1 \le j_0 \le m$, we can pick some $b \in \mathbf{C}$ so that $b^{n_j} \overline{b}^{k_j} - b^{n_{j0}} \overline{b}^{k_{j0}} = 0$ if and only if $n_j = n_{j0}$ and $k_j = k_{j0}$. (Take $|b| \neq 1$ and $\arg b/2\pi$ irrational.) Then $\ell(z) = h(bz) - b^{n_{j0}} \overline{b}^{k_{j0}} h(z)$ is a polynomial in V_f whose nontrivial monomials are the same as those of $h(z)$ except the monomial $z^{n_{j0}} \overline{z}^{k_{j0}}$ whose coefficient in $\ell(z)$ is 0. By repeating this procedure we can eliminate from h any collection of its monomials, and it follows that once a polynomial is in V_f each of its monomials is in V_f as well. So $z^n \overline{z}^k \in V_f$ and $(n,k) \in \sigma(f)$ as required.

Let E_n denote the space of polynomials in z of degree $\le n$, and let E_{∞} denote the space of entire functions.

Proposition 3. *Suppose $f \in C^\infty(\mathbf{C})$ and $\sigma(f) \subseteq [n,k]$. If k is finite then $f = \sum_{s=1}^k h_s \overline{z}^s$ where $h_s \in E_n$ for all $s \le k$. If n is finite then $f = \sum_{r=1}^n h_r z^r$ where $\overline{h}_r \in E_n$ for all $r \le n$. In particular, if $D^{0,k+1} f \equiv 0$ then $f = \sum_{s=1}^k h_s \overline{z}^s$ where $h_s \in E_\infty$ for all $s \le k$.*

Proof. We prove this by induction on k. If $k = 0$ then by Proposition 2, $D^{0,1} f \equiv 0$ implying that $f \in E_\infty$. If $n < \infty$ then $D^{n+1,0} f \equiv 0$ and $f \in E_n$, which proves the case $k = 0$. Assume that our claim holds for $k \le k_0 - 1$ and suppose that $\sigma(f) \subseteq [n, k_0]$. Then $\sigma(D^{0,1} f) \subseteq [n, k_0 - 1]$. It follows that $D^{0,1} f = \sum a_{r,s} z^r \overline{z}^s$, $(r,s) \le (n, k_0 - 1)$. Set $g = \sum (1/(s+1)) a_{r,s} z^r \overline{z}^{s+1}$, $(r,s) \le (n, k_0 - 1)$. Then both f and g are primitive functions (with respect to $D^{0,1}$) of $D^{0,1} f$ and thus $h_0(z) = f(z) - g(z) \in E_\infty$. If $n < \infty$ then $D^{n+1,0} f = D^{n+1,0} g \equiv 0$ and thus $h_0 \in E_n$. So $f(z) = h_0(z) + g(z)$ which is of the desired form. We observe that $D^{0,k+1} f \equiv 0$ implies that $\sigma(f) \subseteq [\infty, k]$ which completes the proof.

Theorem 1 follows easily from the following

Theorem 2. *Let $J \subset M$ be an order ideal and let $B \subset C(\mathbf{C})$ be a closed affine invariant linear subspace. Then:*

(i) *A_J is affine invariant.*
(ii) *$\sigma(B)$ is an order ideal in M.*
(iii) *$B = A_{\sigma(B)}$ and $\sigma(A_J) = J$.*

Proof. (i) We have to show that A_J is affine invariant. Let $f(z) = z^n \overline{z}^k \in A_J$ (i.e., $(n,k) \in J$) and let $a, b \in \mathbf{C}$. Then

$$f(az+b) = (az+b)^n (\overline{az} + b)^k = \sum a_{(u,v)} z^u \overline{z}^v$$

with $(u,v) \le (n,k)$. Thus, as J is an ideal, $(u,v) \in J$ for all the pairs (u,v) which appear in this sum, i.e., $f(az+b) \in A_J$. Hence for each generator $f(z)$ of A_J $f(az+b) \in A_J$, and it follows that A_J is affine invariant.

(ii) We must show that $\sigma(B)$ is an order ideal. By definition $\sigma(B) = \bigcup \{\sigma(f) \colon f \in B\}$ so, by Proposition 1(i) it suffices to show that $\sigma(f)$ is an order ideal for every f in $C(\mathbf{C})$. Let $(n,k) \in \sigma(f)$ i.e. $g = z^n \overline{z}^k \in V_f$. Hence $V_g \subset V_f$ and $\sigma(g) \subset \sigma(f)$. By Proposition 2 $\sigma(g) = [(n,k)]$ and it follows that $\sigma(f)$ is an order ideal.

(iii) $A = A_{\sigma(A)}$. Obviously $A_{\sigma(A)} \subset A$. The inclusion $A \subset A_{\sigma(A)}$ can be obtained by applying Schwartz's theorem on mean periodic functions [S]. Our proof, however, is elementary and does not depend on this result. Note first that it suffices to show that for each $f \in C^\infty(\mathbf{C})$, $f \in A_{\sigma(f)}$. Indeed, from this it will follow that for each f in $A \cap C^\infty(\mathbf{C})$, $f \in A_{\sigma(A)}$, i.e., $A_{\sigma(A)} \supset A \cap C^\infty(\mathbf{C})$ and since $A \cap C^\infty(\mathbf{C})$ is dense in A and $A_{\sigma(A)}$ is closed we obtain that $A_{\sigma(A)} \supset A$. We prove now that if $f \in C^\infty(\mathbf{C})$ then $f \in A_{\sigma(f)}$, by induction on $m =$ the index of $\sigma(f) =$ the cardinality of $\sup \sigma(f)$. The result is trivial if $\sigma(f) = M^*$, so we may assume that $\sigma(f) = \bigcup_{j=1}^{m}[n_j, k_j]$ with either n_j or k_j finite for each j. We shall show that

$$(*) \qquad \text{if } \sigma(f) = \bigcup_{j=1}^{m}[n_j, k_j] \text{ then } f = \sum_{j=1}^{m} f_j \text{ with } f_j \in A_{[n_j, k_j]}.$$

The case $m = 1$ follows from Proposition 3. Assume that $(*)$ holds for all f with the index of $\sigma(f) \le m-1$, and let $\operatorname{index}\sigma(f) = m$. So $\sigma(f) = \bigcup_{j=1}^{m}[n_j, k_j]$ where $\sup\sigma(f) = \{(n_j, k_j)\}_{j=1}^{m}$ and

$$0 \le n_1 < n_2 < \cdots < n_{m-1} < n_m \le \infty, \qquad \infty \ge k_1 > k_2 > \cdots > k_{m-1} > k_m \ge 0.$$

Set $g = D^{0,k_m+1}f$.

We use the following notation:

Let J be an order ideal and let $(r,s) \in M$. Set $J - (r,s) = \{(n-r, k-s) : (n,k) \in J,\ (r,s) \le (n,k)\}$. So $J - (r,s) = \varnothing$ if $(r,s) \notin J$. If $J = \bigcup_{j=1}^{m}[n_j, k_j]$ then $J - (r,s) = \bigcup[n_j - r, k_j - s]$ where the union is taken over all j such that $(r,s) \le (n_j, k_j)$. It follows that $\sigma(g) = \sigma(D^{0,k_m+1}f) = \sigma(f) - (0, k_m+1) = \bigcup_{j=1}^{m-1}[n_j, k_j - k_m - 1]$ (Since $[n_m, k_m - k_m - 1] = [n_m, -1] = \varnothing$.) So, by the induction hypothesis $g = \sum_{j=1}^{m-1} g_j$ with $g_j \in A_{[n_j, k_j - k_m - 1]}$, and by Proposition 2

$$g_j = \sum a_{r,s}^{j} z^r \overline{z}^s \qquad \text{with } (r,s) \le (n_j, k_j - k_m - 1).$$

Set $G_j = \sum((s + k_m + 1)!)/s!\,a_{r,s}^{j} z^r \overline{z}^{s+k_m+1}$ $((r,s) \le (n_j, k_j - k_m - 1))$. Then $G_j \in A_{[n_j, k_j]}$, $1 \le j \le m$. Set $G = \sum_{j=1}^{m-1} G_j$. Then we have

$$D^{0,k_m+1}(f - G) = D^{0,k_m+1}f - D^{0,k_m+1}G = g - g = 0.$$

It follows from Proposition 3 that $f = G + \sum_{i=0}^{k_m} h_i(z)\overline{z}^i = G + G_m$, where $h_i \in E_\infty$, and $G_m = \sum_{i=0}^{k_m} h_i(z)\overline{z}^i$. If n_m is finite we have

$$0 = D^{n_m+1,0}f = D^{n_m+1,0}G + D^{n_m+1,0}G_m = 0 + \sum_{i=0}^{k_m} \overline{z}^i D^{n_m+1,0}h_i(z)$$

and it follows that $D^{n_m+1,0}h_i(z) = 0$ for $0 \le i \le k_m$. Thus $h_i \in E_{n_m}$ for $0 \le i \le k_m$ and $G_m \in A_{[n_m, k_m]}$. Hence $f = \sum_{j=1}^{m} G_j$ where $G_j \in A_{[n_j, k_j]}$ and we are done. The fact that $\sigma(A_J) = J$ follows from the explicit representation of the elements of A_J obtained in Proposition 3 and in $(*)$. This proves Theorem 2.

Proof of Corollary 1. Clearly, the five mentioned spaces are affine and composition invariant. Let A be an affine and composition invariant closed linear subspace of $C(\mathbf{C})$. Set $\lambda(A) = \sup\{k : (n,k) \in \sigma(A)\}$.

(i) If $\lambda(A) = 0$ then the elements of A are analytic. If A is not $A_{[1,0]}$ or the constants then by Theorem 1 $(2,0) \in \sigma(A)$. Applying the composition invariance we conclude that $(n,0) \in \sigma(A)$ for all $n \in \mathbf{N}$ and hence $A = A_{[\infty,0]}$.

(ii) If $\lambda(A) = 1$ then $\overline{z} \in A$ and by the composition invariance $\overline{z} \circ \overline{z} = \overline{\overline{z}} = z \in A$. So $A \supset A_{[1,1]}$. If $(n, k) \in \sigma(A)$ for some $n \geq 2$ then by composition with $\overline{z}$ $(k, n) \in \sigma(A)$ too. Thus $n \leq \lambda(A) = 1$ i.e. $A \subset A_{[1,1]}$.

(iii) If $\lambda(A) \geq 2$ then by applying compositions with $\overline{z}^2$ we conclude that

$$(n, k) \in \sigma(A) \text{ for all } n, k \in \mathbf{N}, \text{ i.e., } A = A_{[\infty,\infty]} = C(\mathbf{C}).$$

Proof of Corollary 2. Let A_h denote the closed affine and composition invariant subspace of $C(\mathbf{C})$ generated by h. It is easy to verify that the elements of A_h operate on B. By Corollary 1, if h is nonlinear (i.e. $h \notin A_{[(1,1)]}$) then A_h is either E_∞ or $C(\mathbf{C})$. Thus $z^2 \in A_h$ and B is an algebra. If h is nonanalytic then A_h is $A_{[(1,1)]}$ or $C(\mathbf{C})$, hence $\overline{z} \in A_h$ and B is self-adjoint.

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