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ZERO CYCLES ON QUADRIC HYPERSURFACES

RICHARD G. SWAN

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ABSTRACT. Let X be a projective quadric hypersurface over a field of characteristic not 2. It is shown that the Chow group $A_0(X)$ of 0-cycles modulo rational equivalence is infinite cyclic, generated by any point of minimal degree.

Let k be a field of characteristic not 2 and let $X \subset \mathbb{P}_k^{d+1}$ be a quadric hypersurface defined by an equation $q = 0$ where q is a quadratic form in $d+2$ variables over k. In [4] I computed the K-theory of X assuming that q is nondegenerate. However the problem of computing the Chow groups of X, which was proposed in [3] is, to the best of my knowledge, still open. I will treat here the first nontrivial case by determining the Chow group $A_0(X)$ of 0-cycles modulo rational equivalence [1]. The result turns out to hold also in the singular case. My original proof made use of the results of [4]. I would like to thank Mohan Kumar for pointing out that this was not necessary and that only elementary facts about A_0 are needed.

Theorem. *Let $X \subset \mathbb{P}_k^{d+1}$ be defined by $q = 0$ where q is a quadratic form over a field k of characteristic not 2 and $d > 0$. Then $A_0(X) = \mathbb{Z}$. It is generated by any rational point if one exists. If X has no rational point then $A_0(X)$ is generated by any point of degree 2 over k.*

We can obviously assume that q is not identically 0 so that $\dim X = d$. If $d = 0$, it is then clear that $A_0(X) = \mathbb{Z} \times \mathbb{Z}$ if X consists of two rational points and $A_0(X) = \mathbb{Z}$ otherwise. For the reader's convenience I will restate the following standard classification for the case $d = 1$.

Lemma 1. *Let $X \subset \mathbb{P}^2$ be a quadric curve. Then one of the following holds:*

(1) *$X = \mathbb{P}^1$ embedded in $\mathbb{P}^2$ by the Veronese embedding.*
(2) *X is a smooth conic with no rational point.*
(3) *$X = L_1 \cup L_2$ is a union of two lines defined over k with one common point.*

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(4) $X = L \cup L'$ *where* L *is a line defined over a quadratic extension of* k (*but not over* k) *and* L' *is its conjugate. The only rational point on* X *is the intersection* $L \cap L'$.

(5) X *is a double line defined by* $l^2 = 0$ *where* l *is a linear form over* k.

If q is nondegenerate and nonisotropic we have case (2). In the isotropic case we can write $q = xy - z^2$, getting case (1). If q reduces to $ax^2 + by^2$ with $ab \neq 0$, we have case (3) if $-a^{-1}b$ is a square in k, and case (4) otherwise. If q reduces to ax^2 we have case (5).

I will write $x \sim y$ when x and y are rationally equivalent. In the following lemmas, X will always denote the quadric hypersurface of the theorem.

Lemma 2. *If x and y are rational points of x then* $x \sim y$.

Proof. Let P be a 2-plane in $\mathbb{P}^{d+1}$ containing the points x and y. Then $P \cong \mathbb{P}^2$. If $P \subset X$ the result is clear. Otherwise $X \cap P$ is as in Lemma 1. In case (1) the result is again clear. Case (2) cannot occur. In case (3) x and y are rationally equivalent to the common point $a = L_1 \cap L_2$. In case (4) there is only one rational point, so $x = y$. Finally, in case (5) x and y lie on the line $l = 0$, so $x \sim y$.

Lemma 3. *The theorem is true if* X *has a rational point*.

Proof. Let x be a rational point and let y be any point. Let $k' = \kappa(y)$ be the residue field of y and let $X' = k' \otimes_k X$ with projection $\pi: X' \to X$. Then X' is just the quadric hypersurface defined by $q = 0$ over k'. Now $\pi^{-1}(y) = \operatorname{Spec} k' \otimes_k \kappa(y)$ has a rational point y' and $\pi^{-1}(x) = \operatorname{Spec} k' \otimes_k \kappa(x) = \operatorname{Spec} k' = \{x'\}$ with x' rational. By Lemma 2, $x' \sim y'$. Therefore $y = \pi_*(y') \sim \pi_*(x') = |k' : k| x$. Thus $A_0(X)$ is geneated by x. Since $\deg: A_0(X) \to \mathbb{Z}$ by $\deg z = |\kappa(z) : k|$, the result follows.

We can now assume that X has no rational point. Since we can write $q = \sum a_i x_i^2$, it is clear that X has points of degree 2. Also X must be smooth since all a_i must be nonzero. The following is a special case of [3, Lemma 13.4].

Lemma 4. *If* X *has no rational point the all points of X have even degree.*

Proof. Suppose $x \in X$ has odd degree. Let $k' = \kappa(x)$. Then $k' \otimes_k X$ has a rational point so that q is isotropic over k'. Since $|k' : k|$ is odd, a theorem of Springer [2, Chap. 7, Theorem 2.3] implies that q is isotropic over k, so X has a rational point.

Lemma 5. *Let* K *be the kernel of* $\deg: A_0(X) \to \mathbb{Z}$. *Then* $2K = 0$.

Proof. We can assume that X has no rational point. Let $x \in X$ have degree 2 and set $k' = \kappa(x)$. Let $X' = k' \otimes_k X$ with projection $\pi: X' \to X$. Then $\pi^{-1}(x) = \operatorname{Spec} k' \otimes_k \kappa(x) = \{x', x''\}$ where x' and x'' are rational over k'. If y is any closed point, $\pi^{-1}(y) = \operatorname{Spec} k' \otimes_k \kappa(y) =$ either $\{y', y''\}$ or $\{z\}$

depending on whether $k' \otimes_k \kappa(y)$ splits or not. By Lemma 3 we can write $y' \sim mx'$ or $z \sim mx'$ getting either $y = \pi_*(y') \sim m\pi_*(x') = mx$ or $2y = \pi_*(z) \sim mx$. It follows that twice any 0-cycle is rationally equivalent to a multiple of x.

Lemma 6. *If x and y have degree 2 then $x \sim y$.*

Proof. We can assume X has no rational point by Lemma 3. Let $V = X(\overline{k})$ be the variety corresponding to X over the algebraic closure $\overline{k}$ of k. The point x corresponds to a pair of points ξ, ξ' of V. Since $\operatorname{char} k \neq 2$, $\kappa(x)$ is Galois over k so that ξ and ξ' are distinct and conjugate over k. The line L spanned by ξ and ξ' is stable under the Galois group and therefore is defined over k. Since $(L \cdot X) = 2$ by Bezout's theorem, we see that $L \cdot X = x$. Similarly, $y = L' \cdot X$ for some line L' defined over k. Since $L \sim L'$ as 1-cycles on $\mathbb{P}^{d+1}$, it follows that $x \sim y$.

If y is a closed point of X I will say that y is "good" if $y \sim mx$ for some integer m and some point x of degree 1 or 2. The theorem will follow from Lemmas 3 and 6 if we can show that all points are good.

Lemma 7. *Let $\pi\colon X' = k' \otimes_k X \to X$ be the canonical projection where $|k' : k|$ is odd. If all points of $\pi^{-1}(y)$ are good, so is y.*

Proof. We can assume that X has no rational points. The same is then true of X' by Springer's theorem as in the proof of Lemma 4. Therefore if x is a point of degree 2 on X then $\pi^{-1}(x) = \{x'\}$, since otherwise $\pi^{-1}(x)$ would consist of two rational points.

Let $k' \otimes_k \kappa(y) = A_1 \times \cdots \times A_r$ where the A_i are local artinian with residue fields $k_i = A_i/\mathfrak{M}_i = \kappa(y_i)$ where the y_i are the points of $\pi^{-1}(y)$. Then $|k' : k| = \dim_{\kappa(y)} k' \otimes_k \kappa(y) = \sum l(A_i)|k_i : \kappa(y)|$ so for some i, $|\kappa(y_i) : \kappa(y)|$ is odd. Since y_i is good, $y_i \sim mx'$ for some m and hence $\pi_*(y_i) = |\kappa(y_i) : \kappa(y)| y \sim m\pi_*(x') = m|k/ : k|x$. Since $|\kappa(y_i) : \kappa(y)|$ is odd and $2y \sim m'x$ for some m' by Lemma 5, the result follows.

Lemma 8. *Let $\eta\colon X' = k' \otimes_k X \to X$ be the canonical projection where $|k' : k| = 2$. Let y be a closed point of X such that $\eta^{-1}(y) = \{y', y''\}$ has two distinct points. If y' is good, so is y.*

Proof. We can assume that X has no rational points. If $x' \in X'$ is rational, then $x = \eta_*(x')$ has degree 2 and $y' \sim mx'$ implies $y = \eta_*(y') \sim mx$. If X' has no rational point and $x \in X$ has degree 2, then $\eta^{-1}(x) = \{x'\}$ where x' has degree 2 on X'. Now $y' \sim mx'$, so $y = \eta_*(y') \sim m\eta_*(x') = 2mx$ as required.

We will now show that all closed points of X are good. Let $y \in X$ have degree n. By induction, we can assume that all points of degree less than n are good on all the hypersurfaces $k' \otimes_k X$. Let K be the "normal closure" of $\kappa(y)$, i.e., the composite of all conjugates of $\kappa(y)$ in the algebraic closure

$\bar{k}$ of k. Let $G = \mathrm{Aut}(K/k)$. Then K^G is purely inseparable over k and therefore of odd degree since $\mathrm{char}\, k \neq 2$. Let H be a 2-Sylow subgroup of G. Then $k' = K^H$ is of odd degree over K^G and hence over k. Let $\pi: X' = k' \otimes_k X \to X$ be the canonical projection. By Lemma 7 it will suffice to show that each point y' of $\pi^{-1}(y)$ is good. Now $\kappa(y')$ is a quotient of $k' \otimes_k \kappa(y)$. Any embedding of $\kappa(y')$ in $\bar{k}$ fixing k' will necessarily send $\kappa(y)$ into K so we get $k' = K^H \subset \kappa(y') \subset K$ and $\kappa(y') = K^L$ for some subgroup L of H. If $L = H$ then y' is rational and hence good. If $L < H$, let M be a maximal proper subgroup of H containing L. Then $|H:M| = 2$. Let $k'' = K^M$ and consider $\eta: X'' = k'' \otimes_k X \to X'$. Since $k'' = K^M \subset K^L = \kappa(y')$ and $|k'':k'| = 2$, $k'' \otimes_{k'} \kappa(y') \approx \kappa(y') \times \kappa(y')$ so $\eta^{-1}(y') = \{u, v\}$ where $\kappa(u) = \kappa(y') = \kappa(v)$. Since $|\kappa(y'):k''| = \frac{1}{2}|\kappa(y'):k'|$, the induction hypothesis shows that u is good. Therefore y' is good by Lemma 8 and hence y is good by Lemma 7.

The corresponding result for the affine case now follows easily.

Corollary. *Let $V \subset \mathbb{A}_k^{d+1}$ be the affine hypersurface defined by $q = 1$ where q is a quadratic form over a field k of characteristic not 2 and $d > 0$. If q is nonisotropic and represents 1 then $A_0(V) = \mathbb{Z}/2\mathbb{Z}$. In all other cases $A_0(V) = 0$.*

Proof. Let $X \subset \mathbb{P}_k^{d+1}$ be defined by $q - y^2 = 0$ and let $X_\infty = X \cap (y = 0)$. Then $V = X - X_\infty$ and we have an exact sequence $A_0(X_\infty) \to A_0(X) \to A_0(V) \to 0$. Since $\deg: A_0(X) \to \mathbb{Z}$ is injective by the theorem, it follows that $A_0(V) = \deg A_0(X)/\deg A_0(X_\infty)$, which is zero unless X has a rational point and X_∞ does not, in which case it is $\mathbb{Z}/2\mathbb{Z}$.

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Department of Mathematics, University of Chicago, Chicago, Illinois 60637