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UNIQUENESS OF APERIODIC KNEADING SEQUENCES

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ABSTRACT. The trapezoidal function $f_e(x)$ is defined for fixed $e \in (0, 1/2)$ by $f_e(x) = (1/e)x$ for $x \in [0,e]$, $f_e(x) = 1$ for $x \in (e, 1-e)$, and $f_e(x) = (1/e)(1-x)$ for $x \in [1-e, 1]$. For a given e and the associated one-parameter family of maps $\{\lambda f_e(x) | \lambda \in [0,1]\}$, we show that if A is an aperiodic kneading sequence, then there is a unique $\lambda \in [0,1]$ so that the itinerary of λ under the map λf_e is A. From this, we conclude that the "stable windows" are dense in $[0,1]$ for the one-parameter family λf_e.

This note is mainly concerned with those maps which are trapezoidal. The trapezoidal function, f_e, is defined for $e \in (0, 1/2)$ by $f_e(x) = x/e$ for $x \in [0, e]$, $f_e(x) = 1$ for $x \in (e, 1-e)$, and $f_e(x) = (1-x)/e$ for $x \in [1-e, 1]$. For a fixed e, one can form a one parameter family of maps by considering the set $\{\lambda f_e | \lambda \in [0,1]\}$.

Throughout this note, the notation and terminology of Beyer, Mauldin, and Stein (BMS) [2] is used. Thus, if g maps $[0,1]$ into itself and $x \in [0,1]$, then the itinerary of x under g is given by $I^g(x) = \{b_i\}_{i \geq 0}$, where $b_i = R$ if $g^i(x) > 1/2$, $b_i = L$ if $g^i(x) < 1/2$, and $b_i = C$ if $g^i(x) = 1/2$ (if $b_i = C$ for some i, then the itinerary stops). We note that $I^g(x)$ is either an infinite sequence of R's and L's or is a finite sequence of R's and L's followed by a C. If g is unimodal and $\lambda \in [0,1]$ is such that the orbit of λ under the scaled map λg contains $1/2$, then the finite sequence $I^{\lambda g}(\lambda)$ is referred to as an MSS sequence [2, 9].

For g unimodal and $\lambda \in [0,1]$, the intinerary of λ under the map λg, $I^{\lambda g}(\lambda)$, is referred to as the kneading sequence of λg [6]. BMS show that $I^{\lambda g}(\lambda)$ is shift maximal in the parity-lexicographical order when g is unimodal and $\lambda \in [0,1]$ (throughout this note, when comparing kneading sequences the parity-lexicographical order is used). Furthermore, if B is a finite or infinite shift maximal sequence, then there is some unimodal map g and some $\lambda \in [0,1]$ so that $I^{\lambda g}(\lambda) = B$. Thus any kneading sequence is shift maximal, and any shift maximal sequence is a kneading sequence. We note that if g is unimodal and

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$\lambda \in [0,1]$, then $I^{\lambda g}(\lambda)$ is either an MSS sequence, or is infinite and periodic, or is infinite and aperiodic. In this note we prove the following theorem.

Theorem A. *Fix $e \in (0,1/2)$. Let A be aperiodic shift maximal sequence. Then there exists a unique $\lambda \in [0,1]$ such that $I^{\lambda f_e}(\lambda) = A$.*

Let us comment on our interest in Theorem A in general terms. For a unimodal map g let

$$\mathscr{P}_g = \{\lambda \in [0,1] | I^{\lambda g}(\lambda) \text{ is an MSS sequence or is infinite and periodic}\}.$$

It is widely believed [7; 4, pp. 31, 69] that $\mathscr{P}_{4x(1-x)}$ is dense in $[0,1]$. We note that $\mathscr{P}_{4x(1-x)}$ consists of precisely those λ such that $\lambda 4x(1-x)$ has a stable orbit [4, p. 69]. Similarly, $\mathscr{P}_{f_e}$ consists of those values of λ such that λf_e has a stable orbit or a finite orbit containing either e or $1-e$. Theorem A and Lemma 1.1 (of Appendix A) imply that $\mathscr{P}_{f_e}$ is dense in $[0,1]$, i.e., the "stable windows" are dense in $[0,1]$. If one could prove Theorem A for the family $\lambda 4x(1-x)$, then again the "stable windows" would be dense in $[0,1]$. Note that there are uncountably many aperiodic shift maximal sequences.

We say that the one-parameter family $\{\lambda g | \lambda \in [0,1]\}$, where g is unimodal, *exhibits uniqueness* provided that for each MSS sequence P there exists exactly one λ such that $I^{\lambda g}(\lambda) = P$. Moreover, the family $\{\lambda g | \lambda \in [0,1]\}$ is said to be *fully unique* if it exhibits uniqueness and if for each aperiodic kneading sequence A there is exactly one λ such that $I^{\lambda g}(\lambda) = A$. Fix $e \in (0,1/2)$. It is known that $\{\lambda f_e | \lambda \in [0,1]\}$ and $\{\lambda 4x(1-x) | \lambda \in [0,1]\}$ exhibit uniqueness [2, 8, 10]. In this note, we establish that the family $\{\lambda f_e\}$ is fully unique. We remark that this is the only one-parameter family shown to be fully unique. If we take $g(x)$ to be $4x(1-x)$, then for certain but not all aperiodic kneading sequences Dennis Sullivan has shown that there is a unique λ so that $I^{\lambda g}(\lambda)$ is the given kneading sequence [11]. Moreover he has shown that if there exists an analytic family that is fully unique, then the family $\lambda 4x(1-x)$ is fully unique [11].

This paper is broken into two sections. Section one consists of general observations, and preliminary comments. Section two contains the proof of Theorem A.

1

BMS [2] show that the one-parameter family λf_e exhibits uniqueness for $0 < e < (3\text{sqrt}(17) - 11)/4$ (we note that $(3\text{sqrt}(17) - 11)/4 \cong 0.3423$). Metropolis and Louck [8] show uniqueness for any $e \in (0,1/2)$. Our proof of Theorem A uses uniqueness for a given e. If $e \in (0,1/2)$ and B is some periodic shift maximal sequence, it is known that there is more than one λ with $I^{\lambda f_e}(\lambda) = B$. This will be discussed presently. Throughout the rest of this note assume that $e \in (0,1/2)$ is fixed. We state the following fact as a theorem and outline its proof in Appendix A.

Theorem B. *Let g be a unimodal Lipschitz continuous concave function that has a continuous derivative in a neighborhood of $x = 1/2$. Furthermore assume that g exhibits uniqueness. Then the following hold.*

(i) *If B is an aperiodic shift maximal sequence, then $\{\lambda \in [0,1] | I^{\lambda g}(\lambda) = B\}$ is a closed interval or consists of a single point.*

(ii) *If $B = \{b_i\}_{i\geq 1}$ is a periodic shift maximal sequence of period k, then exactly one of the following hold.*

(a) *The sequence $b_1 \cdots b_{k-1}C$ is the harmonic of some MSS sequence P, and the set of λ for which $I^{\lambda g}(\lambda) = B$ is an open interval.*

(b) *The sequence $b_1 \cdots b_{k-1}C$ is not the harmonic of some MSS sequence, and the set of λ for which $I^{\lambda g}(\lambda) = B$ is either an open or half open interval. More precisely, if $b_1 \cdots b_{k-1}$ is odd (even), then $(b_1 \cdots b_{k-1}R)^\infty$ $((b_1 \cdots b_{k-1}L)^\infty)$ is a left closed right open interval, and $(b_1 \cdots b_{k-1}L)^\infty$ $\cdot((b_1 \cdots b_{k-1}R)^\infty)$ is an open interval.*

2

Recall that $e \in (0, 1/2)$ is fixed. Set $f = f_e$. For x in $\mathbf{R}$ and $\lambda \in (0, 1]$ set

$$f_{\lambda,R}^{-1}(x) = 1 - (e/\lambda)x,$$

and

$$f_{\lambda,L}^{-1}(x) = (e/\lambda)x.$$

For $P = P_1 \cdots P_n \in \{R, L\}^n$ set

(i) $\rho(P) = \{j | P_j = R\}$, and

(ii) $G_\lambda(P, y) = f_{\lambda,P_1}^{-1}(f_{\lambda,P_2}^{-1}(\cdots(f_{\lambda,P_n}^{-1}(y))\cdots))$.

Beyer and Ebanks [1] have the following theorem, which can be proved by induction on n.

Theorem 2.1. *Let $P = P_1 \cdots P_n$ be a finite sequence of R's and L's. Then,*

$$G_\lambda(P, y) = \sum_{j\in\rho(P)} (-1)^{|\rho(P_1\cdots P_j)|-1}(e/\lambda)^{j-1} + (-1)^{|\rho(P)|}(e/\lambda)^{|P|}y.$$

Remark 2.2. Let $A = \{a_i\}_{i\geq 1}$ be an aperiodic shift maximal sequence, and for each $n \in \mathbf{N}$ let $A|_n = \langle a_1, \ldots, a_n\rangle$. Suppose that $I^{\lambda f}(\lambda) = A$. Observe the following two facts.

(i) For each $n \in \mathbf{N}$, $(\lambda f)^n(\lambda) \notin [e, 1-e]$, for otherwise A would be periodic.

(ii) Fix $n \in \mathbf{N}$. Then $G_\lambda(A|_n, (\lambda f)^n(\lambda)) = \lambda$. To see this note that $\lambda \to (\lambda f)(\lambda) = \lambda f_{a_1}(\lambda) \to (\lambda f)^2(\lambda) = \lambda f_{a_2}(\lambda f_{a_1}(\lambda)) \to \cdots \to (\lambda f)^n(\lambda) = (\lambda f_{a_n}(\cdots(\lambda f_{a_1}(\lambda))\cdots))$, where $f_L(x) = (1/e)x$ and $f_R(x) = (1/e)(1 - x)$, with $(\lambda f)^j(\lambda) \notin [e, 1-e]$ for $1 \leq j \leq n$.

Remark 2.3. For $P = RL^n$, let λ_n be the unique scalar such that $I^{\lambda_n f}(\lambda_n) = P$. BMS show that $\lambda_n \to 1$ as $n \to \infty$. Thus Theorem A holds if A is RL^∞.

Proof of Theorem A. Recall that A is an aperiodic shift maximal sequence. We assume, using Remark 2.3, that $A \neq RL^\infty$. Suppose there is more than one λ such that $I^{\lambda f}(\lambda) = A$. Then Theorem B implies that there exists a closed interval $[\alpha, \beta] \subset [0, 1]$ such that for every $\lambda \in [\alpha, \beta]$ we have $I^{\lambda f}(\lambda) = A$. Note that $\alpha \geq (1 - e)$. Again, say $A = \{a_i\}_{i \geq 1}$ and for each $n \in \mathbf{N}$ let $A|_n$ denote the finite sequence $\langle a_1, \dots, a_n \rangle$.

Let $\lambda \in [\alpha, \beta]$ and $n \in \mathbf{N}$. Then, as in Remark 2.2,

$$G_\lambda(A|_n, (\lambda f)^n(\lambda)) = \lambda. \tag{1}$$

Thus for every $\lambda \in [\alpha, \beta]$ and $n \in \mathbf{N}$ we have that (1) holds.

For $\lambda \in (e, 1]$ and $n \in \mathbf{N}$ set

$$g_{\lambda,n}(x) = G_\lambda(A|_n, (\lambda f)^n(x)), \qquad x \in [0, 1].$$

Then,

$$g_{\lambda,n}(\lambda) = \lambda \qquad \text{for all } \lambda \in [\alpha, \beta] \text{ and } n \in \mathbf{N}.$$

Also, for each $\lambda \in (e, 1]$ set

$$g(\lambda) = \sum_{j \in \rho(A)} (-1)^{|\rho(a_1 \cdots a_j)| - 1} (e/\lambda)^{j-1}.$$

Notice that $g(\lambda)$ is the limit of the $g_{\lambda,n}(\lambda)$ as $n \to \infty$. We have $g(\lambda)$ and $g_{\lambda,n}(x)$ defined for $\lambda \in (e, 1]$, $n \in \mathbf{N}$, and $x \in [0, 1]$.

Next, fix $\lambda \in (e, 1]$, $n \in \mathbf{N}$, and $x \in [0, 1]$. Then,

$$\begin{aligned} g(\lambda) - g_{\lambda,n}(x) = &\sum_{\substack{j \in \rho(A) \\ j > n}} (-1)^{|\rho(a_1 \cdots a_j)|^{-1}} (e/\lambda)^{j-1} \\ &- (-1)^{|\rho(A|_n)|} (e/\lambda)^n (\lambda f)^n(x). \end{aligned}$$

Thus,

$$|g(\lambda) - g_{\lambda,n}(x)| \leq \sum_{\substack{j \in \rho(A) \\ j > n}} ((e/\lambda)^{j-1}) + (e/\lambda)^n \lambda,$$

for $\lambda \in (e, 1]$, $n \in \mathbf{N}$, and $x \in [0, 1]$.

Choose $\gamma < 1$ so that $0 < (e/\lambda) < \gamma$ for $\lambda \in [\alpha, \beta]$. Then for $\lambda \in [\alpha, \beta]$, $n \in \mathbf{N}$, and $x \in [0, 1]$ we have

$$|g(\lambda) - g_{\lambda,n}(x)| \leq \sum_{\substack{j \in \rho(A) \\ j > n}} (\gamma^{j-1}) + \gamma^n \beta.$$

Hence, for every $\varepsilon > 0$ there exists $m \in \mathbf{N}$ such that

$$|g(\lambda) - g_{\lambda,m}(x)| < \varepsilon$$

for all $\lambda \in [\alpha, \beta]$ and $x \in [0, 1]$.

In particular, for every $\varepsilon > 0$ there exists $m \in \mathbf{N}$ so that

$$|g(\lambda) - \lambda| = |g(\lambda) - g_{\lambda,m}(\lambda)| < \varepsilon$$

for every $\lambda \in [\alpha, \beta]$. Thus,

$$g(\lambda) = \lambda$$

on $[\alpha, \beta]$.

However recall that

$$g(\lambda) = \sum_{j \in \rho(A)} (-1)^{|\rho(a_1 \cdots a_j)|-1} (e/\lambda)^{j-1},$$

with $\rho(A)$ infinite. Setting $t = (e/\lambda)$ we find that

$$\left(\sum_{j \in \rho(A)} (-1)^{|\rho(a_1 \cdots a_j)|-1} t^j \right) - e = 0$$

on $[e/\beta, e/\alpha]$. This is a contradiction. Thus we have proven Theorem A.

Appendix A

The following lemmas and theorems are used to prove Theorem B.

Lemma 1.1. *Let f be unimodal, $0 \le \lambda_1 < \lambda_2 \le 1$, and $I^{\lambda_1 f}(\lambda_1)$, $I^{\lambda_2 f}(\lambda_2)$ be distinct elements of $\{R, L\}^{\mathbf{N}}$. Then there is some $\lambda_0 \in (\lambda_1, \lambda_2)$ such that $I^{\lambda_0 f}(\lambda_0)$ is finite.*

Proof. Let k be the first index where $I^{\lambda_1 f}(\lambda_1)$ and $I^{\lambda_2 f}(\lambda_2)$ differ. Set

$$\gamma = \sup\{\lambda \in (\lambda_1, \lambda_2) | I^{\lambda f}(\lambda) \text{ agrees with } I^{\lambda_1 f}(\lambda_1) \text{ in the first } k \text{ positions}\}.$$

The $\lambda_1 < \gamma < \lambda_2$, and $I^{\gamma f}(\gamma)$ is finite, since otherwise the definition of γ is contradicted.

The following two theorems are taken from BMS.

Theorem 1.2. *Let f be a unimodal Lipschitz continuous function that has a continuous derivative in a neighborhood of $x = 1/2$. Suppose $0 \le \lambda_1 < \lambda_2 \le 1$ and A is a shift maximal sequence other than L^∞, C, R^∞, or RL^∞. Suppose further that*

$$I^{\lambda_1 f}(\lambda_1) < A < I^{\lambda_2 f}(\lambda_2).$$

Then there exists some $\lambda \in (\lambda_1, \lambda_2)$ so that $I^{\lambda f}(\lambda) = A$. (The theorem also holds if $I^{\lambda_1 f}(\lambda_1) > A > I^{\lambda_2 f}(\lambda_2)$.)

Theorem 1.3. *Let f be a unimodal Lipschitz continuous concave function with a continuous derivative in a neighborhood of $1/2$. Then for each shift maximal sequence P there is a value of λ so that $I^{\lambda f}(\lambda) = P$.*

In the proof of Theorem 1.2 BMS prove the following lemma.

Lemma 1.4. *Let f be a unimodal Lipschitz continuous function that has a continuous derivative in a neighborhood of $x = 1/2$. Suppose $\lambda_0 \in [0,1]$ is such that $I^{\lambda_0 f}(\lambda_0)$ is finite. Say, $I^{\lambda_0 f}(\lambda_0) = PC$, where $P \in \{R, L\}^n$ for some n. Then there exists an open interval $U \subset [0,1]$ containing λ_0 so that if λ is in U, then*

$$I^{\lambda f}(\lambda) \in \{PC, (PR)^\infty, (PL)^\infty\}.$$

Proof of Theorem B (i). We note that, by Theorem 1.3, there is some λ with $I^{\lambda g}(\lambda) = B$. Let

$$\beta = \sup\{\lambda \in [0,1] | I^{\lambda g}(\lambda) = B\},$$

and

$$\alpha = \inf\{\lambda \in [0,1] | I^{\lambda g}(\lambda) = B\}.$$

Suppose that $\alpha < \beta$. Then Lemma 1.4 implies that both $I^{\alpha g}(\alpha)$ and $I^{\beta g}(\beta)$ are not finite and therefore must both be equal to B. If $I^{\lambda g}(\lambda) = B$ for all $\lambda \in [\alpha, \beta]$, then we are done. Suppose there is some $\lambda \in [\alpha, \beta]$ such that $I^{\lambda g}(\lambda) \neq B$. Then, by Lemma 1.1, there exists some $\lambda_0 \in [\alpha, \beta]$ such that $I^{\lambda_0 g}(\lambda_0) = Q$ is finite.

Case 1. Suppose $Q > B$. Then, using Theorem 1.2, there is some $\lambda > \beta$ so that $Q = I^{\lambda g}(\lambda)$. This contradicts uniqueness.

Case 2. Suppose $Q < B$. The argument is similar to case one.

Remark 1.5. We briefly recall what the harmonics of an MSS sequence are. Let $P \in \{R, L\}^k$ be an MSS sequence; we have temporarily dropped the C. For $n \in \mathbf{N}$, set

$$H_n(P) = \begin{cases} H_{n-1}(P) L H_{n-1}(P), & \text{if } H_{n-1}(P) \text{ is odd}, \\ H_{n-1}(P) R H_{n-1}(P), & \text{if } H_{n-1}(P) \text{ is even}, \end{cases}$$

where $H_0(P) = P$.

Then $H_n(P)$ is called the nth harmonic of P. We let $H_\infty(P)$ be the unique element in $\{R, L\}^{\mathbf{N}}$ that is the common extension of the harmonics of P and note that $H_\infty(P)$ is an aperiodic kneading sequence. Of course, $H_m(P)$ is odd (even) if there is an odd (even) number of R's in $H_m(P)$. The following facts are known [5, 9, 3] (for (ii) see [3, Theorem 3.2, p. 436]).

(i) If P is an MSS sequence and Q is some shift maximal sequence so that $P < Q < H_1(P)$, then $Q = (PR)^\infty$ if $H_1(P) = PRP$ or $Q = (PL)^\infty$ if $H_1(P) = PLP$.

(ii) If $B = \{b_i\}_{i \geq 1}$ is a periodic shift maximal sequence of period k, then $b_1 \cdots b_{k-1} C$ is an MSS sequence. Moreover, if $Q \neq b_1 \cdots b_{k-1}$ is some MSS sequence with $a \in \{R, L\}$ so that $(Qa)^\infty = B$, then Q is the first harmonic of $b_1 \cdots b_{k-1} C$.

Proof of Theorem B (ii). Theorem B (ii) now follows from Remark 1.5, Lemma 1.4, and Theorem 1.2.

REFERENCES

1. W. A. Beyer and B. R. Ebanks, *Quadratic convergence of projections in period doubling for trapezoidal maps*, preprint.
2. W. A. Beyer, R. D. Mauldin, and P. R. Stein, *Shift-maximal sequences in function iteration: existence, uniqueness, and multiplicity*, J. Math. Anal. Appl **115**(1986), 305–362.
3. K. M. Brucks, *MSS sequences, colorings of necklaces, and periodic points of $f(z) = z^2 - 2$*, Adv. in App. Math. **8**(1987), 434–445.
4. P. Collet and J.-P. Eckmann, *Iterated maps on the interval as dynamical systems*, Birkhauser, Basel, 1980.
5. B. Derrida, A. Gervois, and Y. Pomeau, *Iteration of endomorphisms on the real axis and representations of numbers*, Ann. Inst. Henri Poincare **29**(1978), 305–356.
6. R. L. Devaney, *An introduction to chaotic dynamical systems*, Benjamin/Cummings Publishing Co., 1986.
7. C. Grebogi, E. Ott, and J. A. Yorke, *Chaos, strange attractors, and fractal basin boundaries in nonlinear dynamics*, Science **238**(1987), 632–637.
8. J. D. Louck and N. Metropolis, *Symbolic dynamics of trapezoidal maps*, D. Reidel Publishing Company, 1986.
9. N. Metroplois, M. L. Stein, and P. R. Stein, *On finite limit sets for transformations on the unit interval*, J. Combin. Theory **15**(1973), 25–44.
10. J. Milnor and W. Thurston, On iterated maps of the interval, Lect. Notes Math. 1342, Springer, 1988.
11. D. P. Sullivan, personal communications.

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