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# UNIQUENESS OF APERIODIC KNEADING SEQUENCES

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ABSTRACT. The trapezoidal function $f_e(x)$ is defined for fixed $e \in (0, 1/2)$ by $f_e(x) = (1/e)x$ for $x \in [0, e]$, $f_e(x) = 1$ for $x \in (e, 1-e)$, and $f_e(x) = (1/e)(1-x)$ for $x \in [1-e, 1]$. For a given $e$ and the associated one-parameter family of maps $\{\lambda f_e(x) | \lambda \in [0, 1]\}$, we show that if $A$ is an aperiodic kneading sequence, then there is a unique $\lambda \in [0, 1]$ so that the itinerary of $\lambda$ under the map $\lambda f_e$ is $A$. From this, we conclude that the "stable windows" are dense in $[0, 1]$ for the one-parameter family $\lambda f_e$.

This note is mainly concerned with those maps which are trapezoidal. The trapezoidal function, $f_e$, is defined for $e \in (0, 1/2)$ by $f_e(x) = x/e$ for $x \in [0, e]$, $f_e(x) = 1$ for $x \in (e, 1-e)$, and $f_e(x) = (1-x)/e$ for $x \in [1-e, 1]$. For a fixed $e$, one can form a one parameter family of maps by considering the set $\{\lambda f_e | \lambda \in [0, 1]\}$.

Throughout this note, the notation and terminology of Beyer, Mauldin, and Stein (BMS) [2] is used. Thus, if $g$ maps $[0, 1]$ into itself and $x \in [0, 1]$, then the itinerary of $x$ under $g$ is given by $I^g(x) = \{b_i\}_{i \geq 0}$, where $b_i = R$ if $g^i(x) > 1/2$, $b_i = L$ if $g^i(x) < 1/2$, and $b_i = C$ if $g^i(x) = 1/2$ (if $b_i = C$ for some $i$, then the itinerary stops). We note that $I^g(x)$ is either an infinite sequence of $R$'s and $L$'s or is a finite sequence of $R$'s and $L$'s followed by a $C$. If $g$ is unimodal and $\lambda \in [0, 1]$ is such that the orbit of $\lambda$ under the scaled map $\lambda g$ contains $1/2$, then the finite sequence $I^{\lambda g}(\lambda)$ is referred to as an MSS sequence [2, 9].

For $g$ unimodal and $\lambda \in [0, 1]$, the intinerary of $\lambda$ under the map $\lambda g$, $I^{\lambda g}(\lambda)$, is referred to as the kneading sequence of $\lambda g$ [6]. BMS show that $I^{\lambda g}(\lambda)$ is shift maximal in the parity-lexicographical order when $g$ is unimodal and $\lambda \in [0, 1]$ (throughout this note, when comparing kneading sequences the parity-lexicographical order is used). Furthermore, if $B$ is a finite or infinite shift maximal sequence, then there is some unimodal map $g$ and some $\lambda \in [0, 1]$ so that $I^{\lambda g}(\lambda) = B$. Thus any kneading sequence is shift maximal, and any shift maximal sequence is a kneading sequence. We note that if $g$ is unimodal and

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$\lambda \in [0,1]$, then $I^{\lambda g}(\lambda)$ is either an MSS sequence, or is infinite and periodic, or is infinite and aperiodic. In this note we prove the following theorem.

**Theorem A.** *Fix* $e \in (0, 1/2)$. *Let* $A$ *be aperiodic shift maximal sequence. Then there exists a unique* $\lambda \in [0,1]$ *such that* $I^{\lambda f_e}(\lambda) = A$.

Let us comment on our interest in Theorem A in general terms. For a unimodal map $g$ let

$$\mathscr{P}_g = \{\lambda \in [0,1] | I^{\lambda g}(\lambda) \text{ is an MSS sequence or is infinite and periodic}\}.$$

It is widely believed [7; 4, pp. 31, 69] that $\mathscr{P}_{4x(1-x)}$ is dense in $[0,1]$. We note that $\mathscr{P}_{4x(1-x)}$ consists of precisely those $\lambda$ such that $\lambda 4x(1-x)$ has a stable orbit [4, p. 69]. Similarly, $\mathscr{P}_{f_e}$ consists of those values of $\lambda$ such that $\lambda f_e$ has a stable orbit or a finite orbit containing either $e$ or $1-e$. Theorem A and Lemma 1.1 (of Appendix A) imply that $\mathscr{P}_{f_e}$ is dense in $[0,1]$, i.e., the "stable windows" are dense in $[0,1]$. If one could prove Theorem A for the family $\lambda 4x(1-x)$, then again the "stable windows" would be dense in $[0,1]$. Note that there are uncountably many aperiodic shift maximal sequences.

We say that the one-parameter family $\{\lambda g | \lambda \in [0,1]\}$, where $g$ is unimodal, *exhibits uniqueness* provided that for each MSS sequence $P$ there exists exactly one $\lambda$ such that $I^{\lambda g}(\lambda) = P$. Moreover, the family $\{\lambda g | \lambda \in [0,1]\}$ is said to be *fully unique* if it exhibits uniqueness and if for each aperiodic kneading sequence $A$ there is exactly one $\lambda$ such that $I^{\lambda g}(\lambda) = A$. Fix $e \in (0, 1/2)$. It is known that $\{\lambda f_e | \lambda \in [0,1]\}$ and $\{\lambda 4x(1-x) | \lambda \in [0,1]\}$ exhibit uniqueness [2, 8, 10]. In this note, we establish that the family $\{\lambda f_e\}$ is fully unique. We remark that this is the only one-parameter family shown to be fully unique. If we take $g(x)$ to be $4x(1-x)$, then for certain but not all aperiodic kneading sequences Dennis Sullivan has shown that there is a unique $\lambda$ so that $I^{\lambda g}(\lambda)$ is the given kneading sequence [11]. Moreover he has shown that if there exists an analytic family that is fully unique, then the family $\lambda 4x(1-x)$ is fully unique [11].

This paper is broken into two sections. Section one consists of general observations, and preliminary comments. Section two contains the proof of Theorem A.

## 1

BMS [2] show that the one-parameter family $\lambda f_e$ exhibits uniqueness for $0 < e < (3\text{sqrt}(17) - 11)/4$ (we note that $(3\text{sqrt}(17) - 11)/4 \cong 0.3423$). Metropolis and Louck [8] show uniqueness for any $e \in (0, 1/2)$. Our proof of Theorem A uses uniqueness for a given $e$. If $e \in (0, 1/2)$ and $B$ is some periodic shift maximal sequence, it is known that there is more than one $\lambda$ with $I^{\lambda f_e}(\lambda) = B$. This will be discussed presently. Throughout the rest of this note assume that $e \in (0, 1/2)$ is fixed. We state the following fact as a theorem and outline its proof in Appendix $A$.

**Theorem B.** *Let $g$ be a unimodal Lipschitz continuous concave function that has a continuous derivative in a neighborhood of $x = 1/2$. Furthermore assume that $g$ exhibits uniqueness. Then the following hold.*

(i) *If $B$ is an aperiodic shift maximal sequence, then $\{\lambda \in [0,1] | I^{\lambda g}(\lambda) = B\}$ is a closed interval or consists of a single point.*

(ii) *If $B = \{b_i\}_{i\geq 1}$ is a periodic shift maximal sequence of period $k$, then exactly one of the following hold.*

(a) *The sequence $b_1 \cdots b_{k-1}C$ is the harmonic of some MSS sequence $P$, and the set of $\lambda$ for which $I^{\lambda g}(\lambda) = B$ is an open interval.*

(b) *The sequence $b_1 \cdots b_{k-1}C$ is not the harmonic of some MSS sequence, and the set of $\lambda$ for which $I^{\lambda g}(\lambda) = B$ is either an open or half open interval. More precisely, if $b_1 \cdots b_{k-1}$ is odd (even), then $(b_1 \cdots b_{k-1}R)^\infty$ $((b_1 \cdots b_{k-1}L)^\infty)$ is a left closed right open interval, and $(b_1 \cdots b_{k-1}L)^\infty$ $\cdot((b_1 \cdots b_{k-1}R)^\infty)$ is an open interval.*

## 2

Recall that $e \in (0, 1/2)$ is fixed. Set $f = f_e$. For $x$ in $\mathbf{R}$ and $\lambda \in (0,1]$ set

$$f^{-1}_{\lambda,R}(x) = 1 - (e/\lambda)x,$$

and

$$f^{-1}_{\lambda,L}(x) = (e/\lambda)x.$$

For $P = P_1 \cdots P_n \in \{R, L\}^n$ set

(i) $\rho(P) = \{j | P_j = R\}$, and

(ii) $G_\lambda(P, y) = f^{-1}_{\lambda,P_1}(f^{-1}_{\lambda,P_2}(\cdots(f^{-1}_{\lambda,P_n}(y))\cdots))$.

Beyer and Ebanks [1] have the following theorem, which can be proved by induction on $n$.

**Theorem 2.1.** *Let $P = P_1 \cdots P_n$ be a finite sequence of $R$'s and $L$'s. Then,*

$$G_\lambda(P, y) = \sum_{j\in\rho(P)} (-1)^{|\rho(P_1\cdots P_j)|-1}(e/\lambda)^{j-1} + (-1)^{|\rho(P)|}(e/\lambda)^{|P|}y.$$

*Remark* 2.2. Let $A = \{a_i\}_{i\geq 1}$ be an aperiodic shift maximal sequence, and for each $n \in \mathbf{N}$ let $A|_n = \langle a_1, \dots, a_n\rangle$. Suppose that $I^{\lambda f}(\lambda) = A$. Observe the following two facts.

(i) For each $n \in \mathbf{N}$, $(\lambda f)^n(\lambda) \notin [e, 1-e]$, for otherwise $A$ would be periodic.

(ii) Fix $n \in \mathbf{N}$. Then $G_\lambda(A|_n, (\lambda f)^n(\lambda)) = \lambda$. To see this note that $\lambda \to (\lambda f)(\lambda) = \lambda f_{a_1}(\lambda) \to (\lambda f)^2(\lambda) = \lambda f_{a_2}(\lambda f_{a_1}(\lambda)) \to \cdots \to (\lambda f)^n(\lambda) = (\lambda f_{a_n}(\cdots(\lambda f_{a_1}(\lambda))\cdots))$, where $f_L(x) = (1/e)x$ and $f_R(x) = (1/e)(1-x)$, with $(\lambda f)^j(\lambda) \notin [e, 1-e]$ for $1 \leq j \leq n$.

*Remark* 2.3. For $P = RL^n$, let $\lambda_n$ be the unique scalar such that $I^{\lambda_n f}(\lambda_n) = P$. BMS show that $\lambda_n \to 1$ as $n \to \infty$. Thus Theorem A holds if $A$ is $RL^\infty$.

*Proof of Theorem* A. Recall that $A$ is an aperiodic shift maximal sequence. We assume, using Remark 2.3, that $A \neq RL^\infty$. Suppose there is more than one $\lambda$ such that $I^{\lambda f}(\lambda) = A$. Then Theorem B implies that there exists a closed interval $[\alpha, \beta] \subset [0, 1]$ such that for every $\lambda \in [\alpha, \beta]$ we have $I^{\lambda f}(\lambda) = A$. Note that $\alpha \geq (1 - e)$. Again, say $A = \{a_i\}_{i \geq 1}$ and for each $n \in \mathbf{N}$ let $A|_n$ denote the finite sequence $\langle a_1, \ldots, a_n \rangle$.

Let $\lambda \in [\alpha, \beta]$ and $n \in \mathbf{N}$. Then, as in Remark 2.2,

$$G_\lambda(A|_n, (\lambda f)^n(\lambda)) = \lambda. \tag{1}$$

Thus for every $\lambda \in [\alpha, \beta]$ and $n \in \mathbf{N}$ we have that (1) holds.

For $\lambda \in (e, 1]$ and $n \in \mathbf{N}$ set

$$g_{\lambda, n}(x) = G_\lambda(A|_n, (\lambda f)^n(x)), \qquad x \in [0, 1].$$

Then,

$$g_{\lambda, n}(\lambda) = \lambda \qquad \text{for all } \lambda \in [\alpha, \beta] \text{ and } n \in \mathbf{N}.$$

Also, for each $\lambda \in (e, 1]$ set

$$g(\lambda) = \sum_{j \in \rho(A)} (-1)^{|\rho(a_1 \cdots a_j)| - 1} (e/\lambda)^{j-1}.$$

Notice that $g(\lambda)$ is the limit of the $g_{\lambda, n}(\lambda)$ as $n \to \infty$. We have $g(\lambda)$ and $g_{\lambda, n}(x)$ defined for $\lambda \in (e, 1]$, $n \in \mathbf{N}$, and $x \in [0, 1]$.

Next, fix $\lambda \in (e, 1]$, $n \in \mathbf{N}$, and $x \in [0, 1]$. Then,

$$\begin{aligned} g(\lambda) - g_{\lambda, n}(x) = &\sum_{\substack{j \in \rho(A) \\ j > n}} (-1)^{|\rho(a_1 \cdots a_j)|^{-1}} (e/\lambda)^{j-1} \\ &- (-1)^{|\rho(A|_n)|} (e/\lambda)^n (\lambda f)^n(x). \end{aligned}$$

Thus,

$$|g(\lambda) - g_{\lambda, n}(x)| \leq \sum_{\substack{j \in \rho(A) \\ j > n}} ((e/\lambda)^{j-1}) + (e/\lambda)^n \lambda,$$

for $\lambda \in (e, 1]$, $n \in \mathbf{N}$, and $x \in [0, 1]$.

Choose $\gamma < 1$ so that $0 < (e/\lambda) < \gamma$ for $\lambda \in [\alpha, \beta]$. Then for $\lambda \in [\alpha, \beta]$, $n \in \mathbf{N}$, and $x \in [0, 1]$ we have

$$|g(\lambda) - g_{\lambda, n}(x)| \leq \sum_{\substack{j \in \rho(A) \\ j > n}} (\gamma^{j-1}) + \gamma^n \beta.$$

Hence, for every $\varepsilon > 0$ there exists $m \in \mathbf{N}$ such that

$$|g(\lambda) - g_{\lambda, m}(x)| < \varepsilon$$

for all $\lambda \in [\alpha, \beta]$ and $x \in [0, 1]$.

In particular, for every $\varepsilon > 0$ there exists $m \in \mathbf{N}$ so that

$$|g(\lambda) - \lambda| = |g(\lambda) - g_{\lambda,m}(\lambda)| < \varepsilon$$

for every $\lambda \in [\alpha, \beta]$. Thus,

$$g(\lambda) = \lambda$$

on $[\alpha, \beta]$.

However recall that

$$g(\lambda) = \sum_{j \in \rho(A)} (-1)^{|\rho(a_1 \cdots a_j)|-1} (e/\lambda)^{j-1},$$

with $\rho(A)$ infinite. Setting $t = (e/\lambda)$ we find that

$$\left( \sum_{j \in \rho(A)} (-1)^{|\rho(a_1 \cdots a_j)|-1} t^j \right) - e = 0$$

on $[e/\beta, e/\alpha]$. This is a contradiction. Thus we have proven Theorem A.

## Appendix A

The following lemmas and theorems are used to prove Theorem B.

**Lemma 1.1.** *Let $f$ be unimodal, $0 \le \lambda_1 < \lambda_2 \le 1$, and $I^{\lambda_1 f}(\lambda_1)$, $I^{\lambda_2 f}(\lambda_2)$ be distinct elements of $\{R, L\}^{\mathbf{N}}$. Then there is some $\lambda_0 \in (\lambda_1, \lambda_2)$ such that $I^{\lambda_0 f}(\lambda_0)$ is finite.*

*Proof*. Let $k$ be the first index where $I^{\lambda_1 f}(\lambda_1)$ and $I^{\lambda_2 f}(\lambda_2)$ differ. Set

$$\gamma = \sup\{\lambda \in (\lambda_1, \lambda_2) | I^{\lambda f}(\lambda) \text{ agrees with } I^{\lambda_1 f}(\lambda_1) \text{ in the first } k \text{ positions}\}.$$

The $\lambda_1 < \gamma < \lambda_2$, and $I^{\gamma f}(\gamma)$ is finite, since otherwise the definition of $\gamma$ is contradicted.

The following two theorems are taken from BMS.

**Theorem 1.2.** *Let $f$ be a unimodal Lipschitz continuous function that has a continuous derivative in a neighborhood of $x = 1/2$. Suppose $0 \le \lambda_1 < \lambda_2 \le 1$ and $A$ is a shift maximal sequence other than $L^\infty, C, R^\infty$, or $RL^\infty$. Suppose further that*

$$I^{\lambda_1 f}(\lambda_1) < A < I^{\lambda_2 f}(\lambda_2).$$

*Then there exists some $\lambda \in (\lambda_1, \lambda_2)$ so that $I^{\lambda f}(\lambda) = A$. (The theorem also holds if $I^{\lambda_1 f}(\lambda_1) > A > I^{\lambda_2 f}(\lambda_2)$.)*

**Theorem 1.3.** *Let $f$ be a unimodal Lipschitz continuous concave function with a continuous derivative in a neighborhood of $1/2$. Then for each shift maximal sequence $P$ there is a value of $\lambda$ so that $I^{\lambda f}(\lambda) = P$.*

In the proof of Theorem 1.2 BMS prove the following lemma.

**Lemma 1.4.** *Let $f$ be a unimodal Lipschitz continuous function that has a continuous derivative in a neighborhood of $x = 1/2$. Suppose $\lambda_0 \in [0,1]$ is such that $I^{\lambda_0 f}(\lambda_0)$ is finite. Say, $I^{\lambda_0 f}(\lambda_0) = PC$, where $P \in \{R, L\}^n$ for some $n$. Then there exists an open interval $U \subset [0,1]$ containing $\lambda_0$ so that if $\lambda$ is in $U$, then*

$$I^{\lambda f}(\lambda) \in \{PC, (PR)^\infty, (PL)^\infty\}.$$

*Proof of Theorem* B (i). We note that, by Theorem 1.3, there is some $\lambda$ with $I^{\lambda g}(\lambda) = B$. Let

$$\beta = \sup\{\lambda \in [0,1] | I^{\lambda g}(\lambda) = B\},$$

and

$$\alpha = \inf\{\lambda \in [0,1] | I^{\lambda g}(\lambda) = B\}.$$

Suppose that $\alpha < \beta$. Then Lemma 1.4 implies that both $I^{\alpha g}(\alpha)$ and $I^{\beta g}(\beta)$ are not finite and therefore must both be equal to $B$. If $I^{\lambda g}(\lambda) = B$ for all $\lambda \in [\alpha, \beta]$, then we are done. Suppose there is some $\lambda \in [\alpha, \beta]$ such that $I^{\lambda g}(\lambda) \neq B$. Then, by Lemma 1.1, there exists some $\lambda_0 \in [\alpha, \beta]$ such that $I^{\lambda_0 g}(\lambda_0) = Q$ is finite.

*Case* 1. Suppose $Q > B$. Then, using Theorem 1.2, there is some $\lambda > \beta$ so that $Q = I^{\lambda g}(\lambda)$. This contradicts uniqueness.

*Case* 2. Suppose $Q < B$. The argument is similar to case one.

*Remark* 1.5. We briefly recall what the harmonics of an MSS sequence are. Let $P \in \{R, L\}^k$ be an MSS sequence; we have temporarily dropped the $C$. For $n \in \mathbf{N}$, set

$$H_n(P) = \begin{cases} H_{n-1}(P) L H_{n-1}(P), & \text{if } H_{n-1}(P) \text{ is odd}, \\ H_{n-1}(P) R H_{n-1}(P), & \text{if } H_{n-1}(P) \text{ is even}, \end{cases}$$

where $H_0(P) = P$.

Then $H_n(P)$ is called the $n$th harmonic of $P$. We let $H_\infty(P)$ be the unique element in $\{R, L\}^{\mathbf{N}}$ that is the common extension of the harmonics of $P$ and note that $H_\infty(P)$ is an aperiodic kneading sequence. Of course, $H_m(P)$ is odd (even) if there is an odd (even) number of $R$'s in $H_m(P)$. The following facts are known [5, 9, 3] (for (ii) see [3, Theorem 3.2, p. 436]).

(i) If $P$ is an MSS sequence and $Q$ is some shift maximal sequence so that $P < Q < H_1(P)$, then $Q = (PR)^\infty$ if $H_1(P) = PRP$ or $Q = (PL)^\infty$ if $H_1(P) = PLP$.

(ii) If $B = \{b_i\}_{i \geq 1}$ is a periodic shift maximal sequence of period $k$, then $b_1 \cdots b_{k-1} C$ is an MSS sequence. Moreover, if $Q \neq b_1 \cdots b_{k-1}$ is some MSS sequence with $a \in \{R, L\}$ so that $(Qa)^\infty = B$, then $Q$ is the first harmonic of $b_1 \cdots b_{k-1} C$.

*Proof of Theorem* B (ii). Theorem B (ii) now follows from Remark 1.5, Lemma 1.4, and Theorem 1.2.

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