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TWO EXAMPLES OF LOCAL ARTINIAN RINGS

WEIMIN XUE

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ABSTRACT. We answer a question of D. A. Hill in the negative by providing two local artinian rings R and S such that R is right serial but the left indecomposable injective R-module is not uniserial, and that S is not right serial but the left indecomposable injective S-module is uniserial. Moreover R possesses a Morita duality but fails to have self-duality.

A module is *uniserial* in case its submodules are linearly ordered by inclusion. A ring is *left* (*right*) *serial* if it is a direct sum of uniserial left (right) modules. It was shown in [5] that an artinian ring R is right serial if and only if every left indecomposable injective R-module is uniserial, provided R is commutative modulo its radical. Hill [5] asked if this result can be extended to arbitrary artinian rings. The answer is "No". In this note we construct two examples of local artinian rings R and S such that: (1) R is right serial but the left indecomposable injective R-module is not uniserial, and (2) S is not right serial but the left indecomposable injective S-module is uniserial. Moreover the local artinian ring R possesses a (Morita) duality but fails to have self-duality, which solves a question some people are concerned about (see Müller [9, p. 1149], for example). Our examples are based on the profound results of Schofield [10] and Dowbor, Ringel and Simson [4].

Theorem ([10, pp. 214–217] and [4]). *There are division ring extensions* $G \leq F$ *and* $H \leq F$ *with* $\dim({}_GF) = \dim(F_H) = 3$, $\dim(F_G) = \dim({}_HF) = 2$, *and ring isomorphisms* $\alpha: F \cong G$ *and* $\beta: F \cong H$.

Throughout this note the above notations will be fixed.

Example 1. Let $R = F \times F$ as abelian group and define the multiplication via

$$(a, b)(c, d) = (ab, \beta(a)d + bc).$$

Then R is a local artinian ring with $c({}_RR) = 3$ and $c(R_R) = 2$. Hence R is right serial with radical square zero. Let E_R be the right indecomposable injective R-module; then we have

$$E/\mathrm{Soc}(E_R) \cong \mathrm{Hom}_R({}_RJ(R)_R, \mathrm{Soc}(E_R))_R$$

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whose length $= \dim(\mathrm{Hom}_F({}_HF_F, F_F)_H) = \dim(F_H) = 3$. It follows that $c(E_R) = 4$. According to Dlab and Ringel [3, Proposition II.3.3], R is not an exceptional ring. It follows from [3, Proposition II.3.2] that the length of the left indecomposable injective R-module ${}_RU$ is greater than 2. Hence ${}_RU$ is not uniserial.

Azumaya [2] and Morita [7] proved that a right artinian ring has a (Morita) duality if and only if every right indecomposable injective module has finite length. A presentation of duality and self-duality can be found in Anderson and Fuller [1, §23, §24]. We note that the local artinian right serial ring R in Example 1 does possess a duality but does not have self-duality since $c(E_R) \neq c({}_RR)$.

Example 2. Let $T = F \times F$ as abelian group and define the multiplication via

$$(a,b)(c,d) = (ab, \alpha(a)\,d + bc).$$

Then T is a local artinian ring with radical square zero, $c({}_TT) = 4$, and $c(T_T) = 2$. Let V_T be the right indecomposable injective T-module, and $S = \mathrm{End}(V_T)$. Using the same calculation as that of Example 1 we have $c(V_T) = 3$. It follows that ${}_SV_T$ defines a duality. Therefore S is a local artinian ring (the right artinian-ness of S follows from Müller [8, Theorem 10] and a result of [4]) with radical square zero, ${}_SV$ is the left indecomposable injective S-module, $c({}_SV) = c(T_T) = 2$, $c({}_SS) = c(V_T) = 3$, and $T = \mathrm{End}({}_SV)$. So ${}_SV$ is uniserial. Now since $c({}_TT) = 4$, T is not an exceptional ring, neither is S by [3, Proposition III.6.1 and Theorem III.3.1]. According to [3, Proposition II.3.2], $c(S_S) > 2$, since $c({}_SS) = 3$ and $c({}_SV) = 2$. Hence S_S is not uniserial.

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Department of Mathematics, University of Maine, Orono, Maine 04469

Current address: Department of Mathematics, Fujian Normal University, Fuzhou, Fujian 350007, People's Republic of China