THE ISOMETRIES OF $H^1_{\mathcal{H}}$

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ABSTRACT. Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space. In this article we characterize the linear isometries of the Banach space $H^1_{\mathcal{H}}$ onto itself. We show that T is such an isometry iff T is of the form $TF(z) = UF(\psi(z))\psi'(z)$, for $F \in H^1_{\mathcal{H}}$ and z in the unit disc, where ψ is a conformal map of the disc onto itself, and U is a unitary operator on $\mathcal{H}$.

1. INTRODUCTION

Let D denote the open unit disc in the complex plane and E be a finite-dimensional complex Banach space. Then H^p_E stands for the Banach space of all $F: D \to E$ such that $\langle F, e^* \rangle$ belongs to the Hardy class H^p for all $e^* \in E$. The norm on H^p_E is given by

$$\|F\|_p = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \|F(e^{it})\|^p \, dt \right\}^{1/p}, \qquad p < \infty,$$

$$\|F\|_\infty = \operatorname{ess\,sup}\|F(e^{it})\| \quad \left(= \sup_{z \in D} \|F(z)\| \right).$$

(We use the same symbol F to denote the corresponding L^p_E element on the unit circle.) When E is a Hilbert space we write $\mathcal{H}$ for E, and refer to [7] for the properties of $H^p_{\mathcal{H}}$.

The isometries of H^∞ were determined by de Leeuw, Rudin and Wermer [5] and quite independently by Nagasawa [10]. Their results were generalized to the context of $H^\infty_{\mathcal{H}}$ in [1]. In [5] the isometries of H^1 are also described. The method is to use the characterization of the closure of the set of extreme points of the unit ball in H^1 that was established in [4] in order to reduce the problem to the H^∞ case via division by an H^1 function. A complete accounting of these results can be found in the book by Hoffman [8, Chapter 9].

In this article we establish an analogous description of the isometries of $H^1_{\mathcal{H}}$. Our proof, however, requires a quite different approach, since it is known that, when one considers the closure of the set of extreme points of the unit ball,

the situation changes radically as we pass from the scalar to the vector case
[2, Theorem 5]. Moreover, reduction to the H^∞ case via division by an H^1
function is no longer possible if that function is vector valued.

We use principally the characterization of the set of extreme points of the
unit ball in $H^1_{\mathcal{H}}$ (rather than their closure) as given in [2], and the Gleason-
Kahane-Żelazko theorem to establish the following.

Theorem. *Let $\mathcal{H}$ be a complex Hilbert space of finite dimension and let $T: H^1_{\mathcal{H}}$
$\to H^1_{\mathcal{H}}$ be a surjective isometry. Then there exists a conformal map ψ of D
onto D and a fixed unitary operator $U: \mathcal{H} \to \mathcal{H}$ such that for any $F \in H^1_{\mathcal{H}}$
and any $z \in D$,*

$$(*) \qquad\qquad TF(z) = UF(\psi(z))\psi'(z).$$

Since obviously any map of form $(*)$ is a surjective isometry, our theorem
in fact characterizes the isometries of $H^1_{\mathcal{H}}$. When $\mathcal{H}$ is of dimension one,
U of course reduces to a complex number of modulus one, and we have the
scalar result of [5]. The particular conformal map of the disc onto itself given
by $z \to (z - z_0)/(1 - \bar{z}_0 z)$, for some fixed element $z_0 \in D$, will be abbreviated
by B_{z_0}. Throughout §2 $\mathcal{H}$ will denote a complex Hilbert space of fixed finite
dimension n and $\{e_1, \ldots, e_n\}$ is a fixed orthonormal basis of $\mathcal{H}$. Given
$F \in H^1_{\mathcal{H}}$ the coordinate functions f_j are defined by $f_j = \langle F, e_j \rangle$, so that
$F = \sum_{j=1}^n f_j e_j$. ∂D denotes the boundary of D and $A(D)$ is the space of all
complex functions continuous on $\overline{D}$ and analytic on D. Constant functions
are denoted by boldface type and, for $z \in D$, μ_z denotes the unit point mass
concentrated at z.

2. THE ISOMETRIES

Our theorem will be established by means of a sequence of propositions
and lemmas. The first proposition is merely a restatement of [2, Definition
1 and Theorem 2], while the second is a very particular case of the Gleason-
Kahane-Żelazko theorem [11, p. 233]. The third proposition is an elementary
observation.

Proposition 1. *An element $F \in H^1_{\mathcal{H}}$, $F \neq 0$, is not an extreme point of the ball
of radius $\|F\|_1$ if and only if $F = q \cdot G$, where $G \in H^1_{\mathcal{H}}$ and q is a nontrivial
inner function.*

Proposition 2. *Let M be a subspace of $A(D)$ of codimension one which contains
no invertible elements. Then $M = \{f \in A(D): f(z_0) = 0\}$ for some $z_0 \in \overline{D}$.*

Proposition 3. *Let B be a linear space and let A, M be subspaces of B. Then if
$\dim(B/M) = n$ we have $\dim(A/A \cap M) \leq n$. Moreover, if A, B are topological
linear spaces such that the topology on A is stronger than the topology it inherits
from B, and if M is closed in B, then $A \cap M$ is closed in A.*

Proposition 4. *Let V be a complex vector space and let $\{v_1^*, \ldots, v_n^*\}$ be a set of linearly independent functionals on V. Then the space*

$$A := \{(v_1^*(v), v_2^*(v), \ldots, v_n^*(v)): v \in V\}$$

is all of $\mathbf{C}^n$.

Proof. If A were a proper subspace of $\mathbf{C}^n$ then there would exist a nonzero linear map $\Phi: \mathbf{C}^n \to \mathbf{C}$ such that $A \subseteq \ker(\Phi)$. Since for $(a_1, \ldots, a_n) \in A$, $\Phi((a_1, \ldots, a_n)) = \sum_{j=1}^n t_j a_j$, for certain complex numbers t_j not all of which are zero, we would have $\sum_{j=1}^n t_j v_j^* = 0$.

Our principal lemma is the following:

Lemma 1. *Let $T: H^1_{\mathcal{H}} \to H^1_{\mathcal{H}}$ be a surjective linear isometry. Then there is a map $\varphi: D \to D$ and, for each $z \in D$, there is a surjective linear operator $U_0(z): \mathcal{H} \to \mathcal{H}$ such that for $F \in H^1_{\mathcal{H}}$ we have*

$$(1) \qquad\qquad TF(\varphi(z)) = U_0(z)F(z).$$

Proof. Fix $z_0 \in D$ and set $S := \{B_{z_0} \cdot F: F \in H^1_{\mathcal{H}}\}$ and $M = T(S)$. Then S is a closed subspace of $H^1_{\mathcal{H}}$ of codimension n and thus so is M. We let A_n denote the linear subspace of $H^1_{\mathcal{H}}$ which is the set $\{\sum_{j=1}^n f_j e_j: f_j \in A(D)\}$ normed by $\||\sum_{j=1}^n f_j e_j\|| = \max_j\{\|f_j\|_\infty\}$. Then A_n can be identified in a natural way with a function algebra defined on n disjoint copies of the closed unit disc ${}^n\dot{D} := \overline{D} \,\dot{\cup}\, \cdots \,\dot{\cup}\, \overline{D}$ (n summands) and we set $N = M \cap A_n$. By Proposition 3,

   (i) the codimension of N in A_n is not greater than n, and, by Proposition 1, no element $F \in N$ can be an extreme point of the ball in $H^1_{\mathcal{H}}$ of radius $\|F\|_1$ (because this is true of the elements of S and T is an isometry).

Thus, in particular, if $\sum_{j=1}^n f_j e_j \in N$ then

   (ii) for each j the function f_j is not invertible in $A(D)$. (An invertible element of $A(D)$ is necessarily outer [4, p. 469].)

We will show by induction on n that any subspace N of A_n having both properties (i) and (ii) is an L^∞-sum of the form

$$(2) \qquad\qquad N = N_1 \oplus_\infty N_2 \oplus_\infty \cdots \oplus_\infty N_n,$$

where for each j, $1 \le j \le n$, N_j is (isometric to) a subspace of $A(D)$ with codimension 1 (under the obvious map which identifies $f_j e_j$ with f_j).

This fact is trivially true for $n = 1$, so we assume it holds for $n = 1, 2, \ldots, k$ and that N is a subspace of A_{k+1} having properties (i) and (ii). Let $\mu_1, \ldots,$ μ_{k+1} be measures on $\partial^{k+1} D := \partial D \,\dot{\cup}\, \cdots \,\dot{\cup}\, \partial D$ ($k + 1$ summands) such that $N = \bigcap_{j=1}^{k+1} \ker(\mu_j)$. (We do not, of course, assume that the μ_j necessarily constitute a linearly independent set of functionals.) For each j with $1 \le j \le k + 1$ write

$$\mu_j = \mu_j^1 + \mu_j^2,$$

where μ_j^1 is the restriction of μ_j to $\partial^k D$ (the union of the first k circles) and μ_j^2 the restriction to the last circle. We will prove that $\mu_1^1, \mu_2^1, \ldots, \mu_{k+1}^1$ are linearly dependent as elements of $(A_k)^*$.

For if we assume the contrary then, by Proposition 4, we would have that

$$\{(\mu_1^1(F), \mu_2^1(F), \ldots, \mu_{k+1}^1(F)) : F \in A_k\} = \mathbf{C}^{k+1}$$

so that there would exist an $F_0 \in A_k$ with

$$\mu_j^1(F_0) = -\mu_j^2(\mathbf{1}e_{k+1}), \qquad j = 1, \ldots, k+1.$$

Hence $F_0 + \mathbf{1}e_{k+1} \in \bigcap_{j=1}^{k+1} \ker(\mu_j)$ contradicting the hypothesis (ii). This contradiction proves our claim regarding the linear dependence of the functionals $\mu_1^1, \mu_2^1, \ldots, \mu_{k+1}^1$.

Thus, without loss of generality we may assume that $\mu_{k+1}^1 = 0$. (For otherwise, if $\mu_{k+1}^1 = \sum_{j=1}^n \alpha_j \mu_j^1$, we may replace the set $\{\mu_1, \ldots, \mu_{k+1}\}$ by

$$\{\nu_1, \ldots, \nu_{k+1}\}$$

where $\nu_j = \mu_j$ if $j \leq k$ and $\nu_{k+1} = \mu_{k+1} - \sum_{j=1}^n \alpha_j \mu_j$, and note that

$$\bigcap_{j=1}^{k+1} \ker(\nu_j) = \bigcap_{j=1}^{k+1} \ker(\mu_j) = N,$$

and $\nu_{k+1}^1 = 0$.) Hence, making this assumption, we set $N' := \bigcap_{j=1}^k \ker(\mu_j^1)$. Then N' is a subspace of A_k of codimension not greater than k. And we observe that if $\sum_{j=1}^k f_j e_j \in N'$ then $\sum_{j=1}^k f_j e_j + 0e_{k+1} \in N$ so that, since N has property (ii), the same is true of N'. Thus by the inductive hypothesis we have

$$N' = N_1 \oplus_\infty \cdots \oplus_\infty N_k$$

where for each j, $1 \leq j \leq k$, N_j is a subspace of $A(D)$ of codimension one consisting entirely of noninvertible elements. It hence follows that, without loss of generality, we can and do assume that μ_j^1 is supported on the jth circle of $\partial^k D$ and, to end the inductive proof regarding the nature of N, it is enough to show that, for $1 \leq j \leq k$, μ_j^2 is a scalar multiple of $\mu_{k+1}^2 = \mu_{k+1}$.

If we assume this is *not* the case then there would exist a j_0, $1 \leq j_0 \leq k$, and an $f_0 \in A(D)$ such that $\mu_{k+1}^2(f_0 e_{k+1}) = 0$ but $\mu_{j_0}^2(f_0 e_{k+1}) = 1$. Note that $\mu_j^1(\mathbf{1}e_j) \neq 0$ for $1 \leq j \leq n$, (since $\mathbf{1}$ is invertible and $\ker(\mu_j^1) = N_j$) so there are scalars $\alpha_j \in \mathbf{C}$ such that $\mu_j^1(\alpha_j \mathbf{1}e_j) = -\mu_j^2(f_0 e_{k+1})$. Set

$$F = \sum_{j=1}^k \alpha_j \mathbf{1}e_j + f_0 e_{k+1}.$$

We have $F \in \bigcap_{j=1}^{k+1} \ker(\mu_j) = N$ but F does not satisfy (ii) since at least $\alpha_{j_0} \neq 0$. This contradiction completes the proof that N has the form specified in (2).

Thus if $F \in N$, $F = \sum_{j=1}^n f_j e_j$ where, for each j, $f_j \in N_j$, a subspace of $A(D)$ of codimension one consisting entirely of noninvertible elements. Hence, by Proposition 2, for each j there is a unique $w_j \in \overline{D}$ such that $f_j(w_j) = 0$ for all such $F = \sum_{j=1}^n f_j e_j \in N$.

We claim that, in fact, all of the w_j belong to D. For if, say, the first k of the w_j belongs to ∂D and the remainder were points of D, and if we denote by $\overline{N}$ the closure of N in $H^1_{\mathscr{H}}$, then $\overline{N}$ would consist of all elements of the form $\sum_{j=1}^n f_j e_j$, where the f_j are arbitrary elements of H^1 for $1 \leq j \leq k$, and $f_j \in \ker(\mu_{w_j})$ for $j > k$. Thus $\overline{N}$ would be a subspace of $H^1_{\mathscr{H}}$ of codimension $n - k$; whereas $\overline{N} \subseteq M$, a subspace of codimension n.

This contradiction shows that indeed all of the w_j are points of D, and it is then obvious that we must have $w_1 = w_2 = \cdots = w_n = w_0$, for some unique point $w_0 \in D$. For if we had $w_j \neq w_k$ for some j, k with $1 \leq j < k \leq n$ then the element $F = B_{w_j} e_j + B_{w_k} e_k$ would belong to $N \subseteq M$. But M contains no extreme points of the ball of radius $\sqrt{2}$, so that by Proposition 1 F must be divisible by a nontrivial inner function, and this is clearly impossible. Thus $w_j = w_0$ for all j as claimed.

Hence, given a point $z_0 \in D$ determining the subspace S as in the first paragraph of this proof, we define $\varphi(z_0)$ to be this point w_0. Then what we have established is that φ is a map from D to D such that for any $z \in D$ and for any $F \in H^1_{\mathscr{H}}$ with $TF \in A_n$,

(3)   if $F(z) = 0$ then $TF(\varphi(z)) = 0$.

We must show that (3) holds for all $F \in H^1_{\mathscr{H}}$ (not simply for $F \in T^{-1}(A_n)$).

Now for any $z \in D$ we have two linear maps

$$H^1_{\mathscr{H}} \ni F \xrightarrow{V_1} F(z) \in \mathscr{H}$$

and

$$H^1_{\mathscr{H}} \ni F \xrightarrow{V_2} TF(\varphi(z)) \in \mathscr{H}$$

from a Banach space *onto* a finite-dimensional Banach space, which (e.g. by consideration of the Poisson integral), are both easily seen to be continuous. Since, when the functionals are restricted to the dense subspace $A := T^{-1}(A_n)$ of $H^1_{\mathscr{H}}$ we have $\ker(V_1|_A) \subseteq \ker(V_2|_A)$ it follows that $\ker(V_1) \subseteq \ker(V_2)$ so that (3) holds for all $F \in H^1_{\mathscr{H}}$. We thus define, for $e \in \mathscr{H}$, $[U_0(z)](e)$ by $[U_0(z)](e) = V_2(F)$, where F is any element of $H^1_{\mathscr{H}}$ such that $V_2(F) = e$. Then $U_0(z)$ is a well defined linear map from $\mathscr{H}$ to $\mathscr{H}$ and we have

$$V_2 = U_0(z) \circ V_1$$

which establishes that $U_0(z)$ is surjective and, by the definition of V_1 and V_2, that (1) holds.

Lemma 2. *If φ is as in the statement of Lemma 1 then φ is an analytic home-omorphism of the disc onto itself, and if we let $\psi = \varphi^{-1}$ then for $F \in H^1_{\mathscr{H}}$ and $z \in D$ we have*

(4) $$TF(z) = U_0(\psi(z))F(\psi(z)).$$

Proof. By applying Lemma 1 to the isometry T^{-1} we obtain the existence of a map $\psi \colon D \to D$ and, for each $z \in D$, a surjective linear operator $V_0(z) \colon \mathscr{H} \to \mathscr{H}$ such that for $G \in H^1_{\mathscr{H}}$ we have

$$T^{-1}G(\psi(z)) = V_0(z)G(z).$$

Letting $F = T^{-1}G$ we thus have

$$F(\psi(z)) = V_0(z)TF(z)$$

and hence

$$F(\psi \circ \varphi(z)) = V_0(\varphi(z))TF(\varphi(z)) = V_0(\varphi(z))U_0(z)F(z) \qquad \text{(by (1))}$$

for all $F \in H^1_{\mathscr{H}}$ and $z \in D$. Thus, for any $F \in H^1_{\mathscr{H}}$, if $F(z) = 0$ then $F(\psi \circ \varphi(z)) = 0$ from which it follows that $\psi \circ \varphi(z) = z$. An analogous argument obtained by interchanging the roles of T and T^{-1} then gives $\varphi \circ \psi(z) = z$ for $z \in D$ so that φ is a bijective map of D onto itself with inverse ψ. Replacing z by $\psi(z)$ in (1) then gives (4) and it only remains to show that ψ (hence φ) is analytic.

Let the matrix of $U_0(\psi(z))$ with respect to our basis $\{e_1, \ldots, e_n\}$ be

$$\begin{pmatrix} a_{11}(z) \ldots a_{1n}(z) \\ \vdots \qquad\quad \vdots \\ a_{n1}(z) \ldots a_{nn}(z) \end{pmatrix}.$$

If $1 \le j \le n$ and we set $F = \mathbf{e}_j$, we have $TF(z) = \sum_{k=1}^{n} a_{kj}(z)e_k$, so that all entries in the matrix are analytic functions on D. Moreover, for any j with $1 \le j \le n$ and any $z \in D$ we necessarily have

(5) $$\sum_{k=1}^{n} |a_{kj}(z)| > 0,$$

for otherwise $U_0(\psi(z))$ could not be surjective.

Next, for any j with $1 \le j \le n$, if we set $G(z) = ze_j$ we obtain from (4) that

$$TG(z) = \sum_{k=1}^{n} a_{kj}(z)\psi(z)e_k$$

so that (5) then implies that $\psi(z)$ is analytic on D.

The proof of our theorem is then completed by the following:

Lemma 3. *If $U(z)$ is the linear operator mapping $\mathscr{H}$ onto $\mathscr{H}$ defined by $U(z) = U_0(z) \cdot \varphi'(z)$, $z \in D$, then $U(z)$ is equal to a constant unitary operator U. Moreover, for $F \in H^1_{\mathscr{H}}$ we have*

$$TF(z) = UF(\psi(z))\psi'(z), \qquad z \in D.$$

Proof. The map T_1 which sends an element $G \in H^1_{\mathscr{H}}$ to the function $G \circ \varphi(z) \cdot \varphi'(z)$ is a surjective isometry of $H^1_{\mathscr{H}}$ with inverse given by $T_1^{-1}G(z) = G \circ \psi(z) \cdot \psi'(z)$. Now by (4) we have, for $F \in H^1_{\mathscr{H}}$,

$$
\begin{aligned}
T_1 TF(z) &= T_1[U_0(\psi(z))F \circ \psi(z)] \\
&= U_0(\psi \circ \varphi(z))F \circ \psi \circ \varphi(z) \cdot \varphi'(z) \\
&= U(z)F(z)
\end{aligned}
$$

where $U(z) = U_0(z) \cdot \varphi'(z)$. Thus, if we can show that $U(\cdot)$ is a constant isometry of $\mathscr{H}$, we would obtain

$$TF(z) = T_1^{-1}UF(z) = UF(\psi(z)) \cdot \psi'(z)$$

thus completing the proof.

Thus suppose that, with respect to our orthonormal basis $\{e_1, \dots, e_n\}$, for $z \in D$ the matrix of $U(z)$ is

$$
\begin{pmatrix}
u_{11}(z) & \dots & u_{1n}(z) \\
\vdots & & \vdots \\
u_{n1}(z) & \dots & u_{nn}(z)
\end{pmatrix}.
$$

Arguing as in the proof of Lemma 2 we see that each of the entries $u_{ij}(\cdot)$ are H^1 functions, and we use the same symbol u_{ij} to denote the corresponding function on ∂D.

If $f \in H^1$ and $1 \le j \le n$, consider the element $fe_j \in H^1_{\mathscr{H}}$. We have

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|\,dt = \|f\|_1 = \|T_1 T(fe_j)\|_1 = \left\| \sum_{i=1}^{n} f \cdot u_{ij}e_i \right\|_1
$$

$$
= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f|\left(\sum_{i=1}^{n}|u_{ij}|^2\right)^{1/2}dt.
$$

Hence,

$$
0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f|\left[\left(\sum_{i=1}^{n}|u_{ij}|^2\right)^{1/2} - 1\right]dt,
$$

and, since the moduli of H^1 functions are dense in the set of nonnegative, real-valued elements of $L^1(\partial D)$, we conclude that for each j, $1 \le j \le n$,

(6)
$$\sum_{k=1}^{n} |u_{kj}(e^{it})|^2 = 1, \qquad \text{a.e. on } \partial D,$$

which is to say that the column vectors of $U(e^{it})$ are unitary vectors on ∂D for almost all e^{it}. Note that, as a consequence, the entries $u_{kj}(\cdot)$ are not only H^1 elements but, in fact, H^∞ elements.

Since $\sum_{k=1}^{n} |u_{kj}(\cdot)|^2$ is subharmonic on D, we have

(7)
$$\sum_{k=1}^{n} |u_{kj}(z)|^2 \leq 1, \qquad z \in D.$$

Hence, for $z \in D$, the column vectors of $U(z)$ have length not greater than 1. Also if $1 \leq j < m \leq n$ consider, for $f \in H^1$, the element $fe_j + fe_m$ of $H^1_{\mathscr{H}}$. We have

$$\sqrt{2} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|\, dt = \|fe_j + fe_m\|_1 = \|T_1 T(fe_j + fe_m)\|_1$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f| \left(\sum_{k=1}^{n} |u_{kj} + u_{km}|^2 \right)^{1/2} dt.$$

Thus

$$0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f| \left[\left(\sum_{k=1}^{n} |u_{kj} + u_{km}|^2 \right)^{1/2} - \sqrt{2} \right] dt,$$

so again using the density of the moduli of H^1 elements in the set of nonnegative, real-valued elements of $L^1(\partial D)$ we get that $\sum_{k=1}^{n} |u_{kj} + u_{km}|^2 = 2$ a.e. on ∂D. Thus

$$\sum_{k=1}^{n} [|u_{kj}(e^{it})|^2 + |u_{km}(e^{it})|^2 + 2\,\text{Re}(u_{kj}(e^{it})\bar{u}_{km}(e^{it}))] = 2$$

which, together with (6) gives

(8)
$$\sum_{k=1}^{n} \text{Re}(u_{kj}(e^{it})\bar{u}_{km}(e^{it})) = 0 \qquad \text{a.e. on } \partial D.$$

If we replace $fe_j + fe_m$ by $fe_j + ife_m$, the same argument then gives

(9)
$$\sum_{k=1}^{n} \text{Im}(u_{kj}(e^{it})\bar{u}_{km}(e^{it})) = 0 \qquad \text{a.e. on } \partial D$$

so that (8) and (9) together give

$$\sum_{k=1}^{n} u_{kj}(e^{it})\bar{u}_{km}(e^{it}) = 0 \qquad \text{a.e. on } \partial D.$$

We have thus shown that $U(\cdot)$ is made up of column vectors which are of unit length and pairwise orthogonal a.e. on ∂D. That is, $U(e^{it})$ is a.e. a unitary operator on ∂D, and if we denote by $V(e^{it}) = [v_{kj}(e^{it})]$ the operator $U^*(e^{it})$, then the rule for computing the inverse of a matrix shows that the entires $v_{kj}(e^{it})$, considered as functions on ∂D, belong to H^∞ and we use the same symbols $v_{kj}(\cdot)$ to denote the corresponding functions defined on the disc.

An argument analogous to that which produced (7) shows that for each j, $1 \le j \le n$, and each $z \in D$,

$$\sum_{k=1}^{n} |v_{kj}(z)|^2 \le 1$$

and since the rows, as well as the columns of $[v_{kj}(e^{it})]$ have unit length a.e. we have, for $1 \le j \le n$,

(10)
$$\sum_{k=1}^{n} |v_{jk}(z)|^2 \le 1, \qquad z \in D.$$

Thus

$$\sum_{k=1}^{n} v_{jk}(e^{it})u_{km}(e^{it}) = \delta_{jm} \qquad \text{a.e. on } \partial D,$$

and hence

(11)
$$\sum_{k=1}^{n} v_{jk}(z)u_{km}(z) = \delta_{jm} \qquad \text{on all of } D.$$

And since $\sum_{k=1}^n v_{jk}(z)u_{km}(z)$ is equal to the inner product

$$\langle v_{j1}(z)e_1 + \cdots + v_{jn}(z)e_n, \ \bar{u}_{1j}(z)e_1 + \cdots + \bar{u}_{nj}(z)e_n \rangle,$$

(11), together with (7) and (10), shows that $\sum_{k=1}^n |u_{kj}(z)|^2 = 1$ for all $z \in D$.

We have thus shown that, for $1 \le j \le n$, $z \to U(z)e_j$ is a vector-valued function defined on D to $\mathscr{H}$ such that $\|U(z)e_j\| = 1$ for $z \in D$. Hence the strong maximum modulus theorem for analytic vector-valued functions [12, Theorem 3.2] implies that $U(z)e_j$ is constant. Hence $U(\cdot)$ is a constant and the proof is complete.

3. REMARKS AND PROBLEMS

(a) Can one establish the theorem of this article for H^1_E, where E belongs to a class of finite-dimensional Banach spaces properly containing Hilbert space? In [3] necessary and sufficient conditions were obtained on a finite-dimensional complex Banach space E which allow the description of the isometries of $H^\infty_{\mathscr{H}}$ given in [1] to be extended to H^∞_E. Thus, can one similarly characterize those finite-dimensional Banach spaces E which are such that the H^1 theorem established in this article extends to H^1_E? Since the condition on E obtained in the

H^∞ case was that it not admit nontrivial L^∞-summands, it might be tempting to conjecture that the proper H^1 condition is the absence of L^1-summands. This condition is clearly necessary, but is it sufficient?

(b) Quite recently Lin [9] has been able to extend the H^∞ result of [1] to H_E^∞, for certain infinite-dimensional Banach spaces E. Thus does the H^1 theorem of our paper have an analogue for infinite-dimensional range spaces? Quite obviously, many of the arguments used in this article are finite-dimensional in nature.

(c) The isometries of H^p for $1 < p < \infty$, $p \neq 2$, have been described by Forelli [6]. To the best of the authors' knowledge, there is no formulation of Forelli's theorem for vector-valued functions which exists in the literature.

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