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THE ISOMETRIES OF $H^1_{\mathscr{H}}$

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ABSTRACT. Let $\mathscr{H}$ be a finite-dimensional complex Hilbert space. In this article we characterize the linear isometries of the Banach space $H^1_{\mathscr{H}}$ onto itself. We show that T is such an isometry iff T is of the form $TF(z) = UF(\psi(z))\psi'(z)$, for $F \in H^1_{\mathscr{H}}$ and z in the unit disc, where ψ is a conformal map of the disc onto itself, and U is a unitary operator on $\mathscr{H}$.

1. INTRODUCTION

Let D denote the open unit disc in the complex plane and E be a finite-dimensional complex Banach space. Then H^p_E stands for the Banach space of all $F: D \to E$ such that $\langle F, e^*\rangle$ belongs to the Hardy class H^p for all $e^* \in E$. The norm on H^p_E is given by

$$\|F\|_p = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \|F(e^{it})\|^p\, dt\right\}^{1/p}, \qquad p < \infty,$$

$$\|F\|_\infty = \operatorname{ess\,sup}\|F(e^{it})\| \left(= \sup_{z\in D}\|F(z)\|\right).$$

(We use the same symbol F to denote the corresponding L^p_E element on the unit circle.) When E is a Hilbert space we write $\mathscr{H}$ for E, and refer to [7] for the properties of $H^p_{\mathscr{H}}$.

The isometries of H^∞ were determined by de Leeuw, Rudin and Wermer [5] and quite independently by Nagasawa [10]. Their results were generalized to the context of $H^\infty_{\mathscr{H}}$ in [1]. In [5] the isometries of H^1 are also described. The method is to use the characterization of the closure of the set of extreme points of the unit ball in H^1 that was established in [4] in order to reduce the problem to the H^∞ case via division by an H^1 function. A complete accounting of these results can be found in the book by Hoffman [8, Chapter 9].

In this article we establish an analogous description of the isometries of $H^1_{\mathscr{H}}$. Our proof, however, requires a quite different approach, since it is known that, when one considers the closure of the set of extreme points of the unit ball,

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the situation changes radically as we pass from the scalar to the vector case [2, Theorem 5]. Moreover, reduction to the H^∞ case via division by an H^1 function is no longer possible if that function is vector valued.

We use principally the characterization of the set of extreme points of the unit ball in $H^1_{\mathscr{H}}$ (rather than their closure) as given in [2], and the Gleason-Kahane-Żelazko theorem to establish the following.

Theorem. *Let $\mathscr{H}$ be a complex Hilbert space of finite dimension and let $T: H^1_{\mathscr{H}} \to H^1_{\mathscr{H}}$ be a surjective isometry. Then there exists a conformal map ψ of D onto D and a fixed unitary operator $U: \mathscr{H} \to \mathscr{H}$ such that for any $F \in H^1_{\mathscr{H}}$ and any $z \in D$,*

$$(*) \qquad TF(z) = UF(\psi(z))\psi'(z).$$

Since obviously any map of form $(*)$ is a surjective isometry, our theorem in fact characterizes the isometries of $H^1_{\mathscr{H}}$. When $\mathscr{H}$ is of dimension one, U of course reduces to a complex number of modulus one, and we have the scalar result of [5]. The particular conformal map of the disc onto itself given by $z \to (z - z_0)/(1 - \bar{z}_0 z)$, for some fixed element $z_0 \in D$, will be abbreviated by B_{z_0}. Throughout §2 $\mathscr{H}$ will denote a complex Hilbert space of fixed finite dimension n and $\{e_1, \dots, e_n\}$ is a fixed orthonormal basis of $\mathscr{H}$. Given $F \in H^1_{\mathscr{H}}$ the coordinate functions f_j are defined by $f_j = \langle F, e_j \rangle$, so that $F = \sum_{j=1}^n f_j e_j$. ∂D denotes the boundary of D and $A(D)$ is the space of all complex functions continuous on $\overline{D}$ and analytic on D. Constant functions are denoted by boldface type and, for $z \in D$, μ_z denotes the unit point mass concentrated at z.

2. The isometries

Our theorem will be established by means of a sequence of propositions and lemmas. The first proposition is merely a restatement of [2, Definition 1 and Theorem 2], while the second is a very particular case of the Gleason-Kahane-Żelazko theorem [11, p. 233]. The third proposition is an elementary observation.

Proposition 1. *An element $F \in H^1_{\mathscr{H}}$, $F \neq 0$, is not an extreme point of the ball of radius $\|F\|_1$ if and only if $F = q \cdot G$, where $G \in H^1_{\mathscr{H}}$ and q is a nontrivial inner function.*

Proposition 2. *Let M be a subspace of $A(D)$ of codimension one which contains no invertible elements. Then $M = \{f \in A(D) : f(z_0) = 0\}$ for some $z_0 \in \overline{D}$.*

Proposition 3. *Let B be a linear space and let A, M be subspaces of B. Then if $\dim(B/M) = n$ we have $\dim(A/A \cap M) \leq n$. Moreover, if A, B are topological linear spaces such that the topology on A is stronger than the topology it inherits from B, and if M is closed in B, then $A \cap M$ is closed in A.*

Proposition 4. *Let V be a complex vector space and let $\{v_1^*, \ldots, v_n^*\}$ be a set of linearly independent functionals on V. Then the space*

$$A := \{(v_1^*(v), v_2^*(v), \ldots, v_n^*(v)) : v \in V\}$$

is all of $\mathbf{C}^n$.

Proof. If A were a proper subspace of $\mathbf{C}^n$ then there would exist a nonzero linear map $\Phi: \mathbf{C}^n \to \mathbf{C}$ such that $A \subseteq \ker(\Phi)$. Since for $(a_1, \ldots, a_n) \in A$, $\Phi((a_1, \ldots, a_n)) = \sum_{j=1}^n t_j a_j$, for certain complex numbers t_j not all of which are zero, we would have $\sum_{j=1}^n t_j v_j^* = 0$.

Our principal lemma is the following:

Lemma 1. *Let $T: H^1_{\mathscr{H}} \to H^1_{\mathscr{H}}$ be a surjective linear isometry. Then there is a map $\varphi: D \to D$ and, for each $z \in D$, there is a surjective linear operator $U_0(z): \mathscr{H} \to \mathscr{H}$ such that for $F \in H^1_{\mathscr{H}}$ we have*

$$TF(\varphi(z)) = U_0(z)F(z). \tag{1}$$

Proof. Fix $z_0 \in D$ and set $S := \{B_{z_0} \cdot F : F \in H^1_{\mathscr{H}}\}$ and $M = T(S)$. Then S is a closed subspace of $H^1_{\mathscr{H}}$ of codimension n and thus so is M. We let A_n denote the linear subspace of $H^1_{\mathscr{H}}$ which is the set $\{\sum_{j=1}^n f_j e_j : f_j \in A(D)\}$ normed by $||| \sum_{j=1}^n f_j e_j ||| = \max_j\{\|f_j\|_\infty\}$. Then A_n can be identified in a natural way with a function algebra defined on n disjoint copies of the closed unit disc ${}^n\overline{D} := \overline{D} \dot{\cup} \cdots \dot{\cup} \overline{D}$ (n summands) and we set $N = M \cap A_n$. By Proposition 3,

(i) the codimension of N in A_n is not greater than n, and, by Proposition 1, no element $F \in N$ can be an extreme point of the ball in $H^1_{\mathscr{H}}$ of radius $\|F\|_1$ (because this is true of the elements of S and T is an isometry).

Thus, in particular, if $\sum_{j=1}^n f_j e_j \in N$ then

(ii) for each j the function f_j is not invertible in $A(D)$. (An invertible element of $A(D)$ is necessarily outer [4, p. 469].)

We will show by induction on n that any subspace N of A_n having both properties (i) and (ii) is an L^∞-sum of the form

$$N = N_1 \oplus_\infty N_2 \oplus_\infty \cdots \oplus_\infty N_n, \tag{2}$$

where for each j, $1 \le j \le n$, N_j is (isometric to) a subspace of $A(D)$ with codimension 1 (under the obvious map which identifies $f_j e_j$ with f_j).

This fact is trivially true for $n = 1$, so we assume it holds for $n = 1, 2, \ldots, k$ and that N is a subspace of A_{k+1} having properties (i) ana (ii). Let $\mu_1, \ldots,$ μ_{k+1} be measures on $\partial^{k+1}D := \partial D \dot{\cup} \cdots \dot{\cup} \partial D$ ($k+1$ summands) such that $N = \bigcap_{j=1}^{k+1} \ker(\mu_j)$. (We do not, of course, assume that the μ_j necessarily constitute a linearly independent set of functionals.) For each j with $1 \le j \le k+1$ write

$$\mu_j = \mu_j^1 + \mu_j^2,$$

where μ_j^1 is the restriction of μ_j to $\partial^k D$ (the union of the first k circles) and μ_j^2 the restriction to the last circle. We will prove that $\mu_1^1, \mu_2^1, \dots, \mu_{k+1}^1$ are linearly dependent as elements of $(A_k)^*$.

For if we assume the contrary then, by Proposition 4, we would have that

$$\{(\mu_1^1(F), \mu_2^1(F), \dots, \mu_{k+1}^1(F)) \colon F \in A_k\} = \mathbf{C}^{k+1}$$

so that there would exist an $F_0 \in A_k$ with

$$\mu_j^1(F_0) = -\mu_j^2(\mathbf{1}e_{k+1}), \qquad j = 1, \dots, k+1.$$

Hence $F_0 + \mathbf{1}e_{k+1} \in \bigcap_{j=1}^{k+1} \ker(\mu_j)$ contradicting the hypothesis (ii). This contradiction proves our claim regarding the linear dependence of the functionals $\mu_1^1, \mu_2^1, \dots, \mu_{k+1}^1$.

Thus, without loss of generality we may assume that $\mu_{k+1}^1 = 0$. (For otherwise, if $\mu_{k+1}^1 = \sum_{j=1}^n \alpha_j \mu_j^1$, we may replace the set $\{\mu_1, \dots, \mu_{k+1}\}$ by

$$\{\nu_1, \dots, \nu_{k+1}\}$$

where $\nu_j = \mu_j$ if $j \le k$ and $\nu_{k+1} = \mu_{k+1} - \sum_{j=1}^n \alpha_j \mu_j$, and note that

$$\bigcap_{j=1}^{k+1} \ker(\nu_j) = \bigcap_{j=1}^{k+1} \ker(\mu_j) = N,$$

and $\nu_{k+1}^1 = 0$.) Hence, making this assumption, we set $N' := \bigcap_{j=1}^k \ker(\mu_j^1)$. Then N' is a subspace of A_k of codimension not greater than k. And we observe that if $\sum_{j=1}^k f_j e_j \in N'$ then $\sum_{j=1}^k f_j e_j + \mathbf{0}e_{k+1} \in N$ so that, since N has property (ii), the same is true of N'. Thus by the inductive hypothesis we have

$$N' = N_1 \oplus_\infty \cdots \oplus_\infty N_k$$

where for each j, $1 \le j \le k$, N_j is a subspace of $A(D)$ of codimension one consisting entirely of noninvertible elements. It hence follows that, without loss of generality, we can and do assume that μ_j^1 is supported on the jth circle of $\partial^k D$ and, to end the inductive proof regarding the nature of N, it is enough to show that, for $1 \le j \le k$, μ_j^2 is a scalar multiple of $\mu_{k+1}^2 = \mu_{k+1}$.

If we assume this is *not* the case then there would exist a j_0, $1 \le j_0 \le k$, and an $f_0 \in A(D)$ such that $\mu_{k+1}^2(f_0 e_{k+1}) = 0$ but $\mu_{j_0}^2(f_0 e_{k+1}) = 1$. Note that $\mu_j^1(\mathbf{1}e_j) \ne 0$ for $1 \le j \le n$, (since $\mathbf{1}$ is invertible and $\ker(\mu_j^1) = N_j$) so there are scalars $\alpha_j \in \mathbf{C}$ such that $\mu_j^1(\alpha_j \mathbf{1}e_j) = -\mu_j^2(f_0 e_{k+1})$. Set

$$F = \sum_{j=1}^k \alpha_j \mathbf{1}e_j + f_0 e_{k+1}.$$

We have $F \in \bigcap_{j=1}^{k+1} \ker(\mu_j) = N$ but F does not satisfy (ii) since at least $\alpha_{j_0} \neq 0$. This contradiction completes the proof that N has the form specified in (2).

Thus if $F \in N$, $F = \sum_{j=1}^n f_j e_j$ where, for each j, $f_j \in N_j$, a subspace of $A(D)$ of codimension one consisting entirely of noninvertible elements. Hence, by Proposition 2, for each j there is a unique $w_j \in \overline{D}$ such that $f_j(w_j) = 0$ for all such $F = \sum_{j=1}^n f_j e_j \in N$.

We claim that, in fact, all of the w_j belong to D. For if, say, the first k of the w_j belongs to ∂D and the remainder were points of D, and if we denote by $\overline{N}$ the closure of N in $H^1_{\mathscr{H}}$, then $\overline{N}$ would consist of all elements of the form $\sum_{j=1}^n f_j e_j$, where the f_j are arbitrary elements of H^1 for $1 \leq j \leq k$, and $f_j \in \ker(\mu_{w_j})$ for $j > k$. Thus $\overline{N}$ would be a subspace of $H^1_{\mathscr{H}}$ of codimension $n - k$; whereas $\overline{N} \subseteq M$, a subspace of codimension n.

This contradiction shows that indeed all of the w_j are points of D, and it is then obvious that we must have $w_1 = w_2 = \cdots = w_n = w_0$, for some unique point $w_0 \in D$. For if we had $w_j \neq w_k$ for some j, k with $1 \leq j < k \leq n$ then the element $F = B_{w_j} e_j + B_{w_k} e_k$ would belong to $N \subseteq M$. But M contains no extreme points of the ball of radius $\sqrt{2}$, so that by Proposition 1 F must be divisible by a nontrivial inner function, and this is clearly impossible. Thus $w_j = w_0$ for all j as claimed.

Hence, given a point $z_0 \in D$ determining the subspace S as in the first paragraph of this proof, we define $\varphi(z_0)$ to be this point w_0. Then what we have established is that φ is a map from D to D such that for any $z \in D$ and for any $F \in H^1_{\mathscr{H}}$ with $TF \in A_n$,

$$\text{(3)} \qquad \text{if } F(z) = 0 \quad \text{then} \quad TF(\varphi(z)) = 0.$$

We must show that (3) holds for all $F \in H^1_{\mathscr{H}}$ (not simply for $F \in T^{-1}(A_n)$).

Now for any $z \in D$ we have two linear maps

$$H^1_{\mathscr{H}} \ni F \xrightarrow{V_1} F(z) \in \mathscr{H}$$

and

$$H^1_{\mathscr{H}} \ni F \xrightarrow{V_2} TF(\varphi(z)) \in \mathscr{H}$$

from a Banach space *onto* a finite-dimensional Banach space, which (e.g. by consideration of the Poisson integral), are both easily seen to be continuous. Since, when the functionals are restricted to the dense subspace $A := T^{-1}(A_n)$ of $H^1_{\mathscr{H}}$ we have $\ker(V_1|_A) \subseteq \ker(V_2|_A)$ it follows that $\ker(V_1) \subseteq \ker(V_2)$ so that (3) holds for all $F \in H^1_{\mathscr{H}}$. We thus define, for $e \in \mathscr{H}$, $[U_0(z)](e)$ by $[U_0(z)](e) = V_2(F)$, where F is any element of $H^1_{\mathscr{H}}$ such that $V_2(F) = e$. Then $U_0(z)$ is a well defined linear map from $\mathscr{H}$ to $\mathscr{H}$ and we have

$$V_2 = U_0(z) \circ V_1$$

which establishes that $U_0(z)$ is surjective and, by the definition of V_1 and V_2, that (1) holds.

Lemma 2. *If φ is as in the statement of Lemma 1 then φ is an analytic homeomorphism of the disc onto itself, and if we let $\psi = \varphi^{-1}$ then for $F \in H^1_{\mathscr{H}}$ and $z \in D$ we have*

$$TF(z) = U_0(\psi(z))F(\psi(z)). \tag{4}$$

Proof. By applying Lemma 1 to the isometry T^{-1} we obtain the existence of a map $\psi: D \to D$ and, for each $z \in D$, a surjective linear operator $V_0(z): \mathscr{H} \to \mathscr{H}$ such that for $G \in H^1_{\mathscr{H}}$ we have

$$T^{-1}G(\psi(z)) = V_0(z)G(z).$$

Letting $F = T^{-1}G$ we thus have

$$F(\psi(z)) = V_0(z)TF(z)$$

and hence

$$F(\psi \circ \varphi(z)) = V_0(\varphi(z))TF(\varphi(z)) = V_0(\varphi(z))U_0(z)F(z) \qquad \text{(by (1))}$$

for all $F \in H^1_{\mathscr{H}}$ and $z \in D$. Thus, for any $F \in H^1_{\mathscr{H}}$, if $F(z) = 0$ then $F(\psi \circ \varphi(z)) = 0$ from which it follows that $\psi \circ \varphi(z) = z$. An analogous argument obtained by interchanging the roles of T and T^{-1} then gives $\varphi \circ \psi(z) = z$ for $z \in D$ so that φ is a bijective map of D onto itself with inverse ψ. Replacing z by $\psi(z)$ in (1) then gives (4) and it only remains to show that ψ (hence φ) is analytic.

Let the matrix of $U_0(\psi(z))$ with respect to our basis $\{e_1, \dots, e_n\}$ be

$$\begin{pmatrix} a_{11}(z) & \dots & a_{1n}(z) \\ \vdots & & \vdots \\ a_{n1}(z) & \dots & a_{nn}(z) \end{pmatrix}.$$

If $1 \le j \le n$ and we set $F = \mathbf{e}_j$, we have $TF(z) = \sum_{k=1}^n a_{kj}(z)e_k$, so that all entries in the matrix are analytic functions on D. Moreover, for any j with $1 \le j \le n$ and any $z \in D$ we necessarily have

$$\sum_{k=1}^{n} |a_{kj}(z)| > 0, \tag{5}$$

for otherwise $U_0(\psi(z))$ could not be surjective.

Next, for any j with $1 \le j \le n$, if we set $G(z) = ze_j$ we obtain from (4) that

$$TG(z) = \sum_{k=1}^{n} a_{kj}(z)\psi(z)e_k$$

so that (5) then implies that $\psi(z)$ is analytic on D.

The proof of our theorem is then completed by the following:

Lemma 3. *If $U(z)$ is the linear operator mapping $\mathscr{H}$ onto $\mathscr{H}$ defined by $U(z) = U_0(z)\cdot\varphi'(z)$, $z \in D$, then $U(z)$ is equal to a constant unitary operator U. Moreover, for $F \in H^1_{\mathscr{H}}$ we have*

$$TF(z) = UF(\psi(z))\psi'(z), \qquad z \in D.$$

Proof. The map T_1 which sends an element $G \in H^1_{\mathscr{H}}$ to the function $G\circ\varphi(z)\cdot\varphi'(z)$ is a surjective isometry of $H^1_{\mathscr{H}}$ with inverse given by $T_1^{-1}G(z) = G\circ\psi(z)\cdot\psi'(z)$. Now by (4) we have, for $F \in H^1_{\mathscr{H}}$,

$$\begin{aligned} T_1TF(z) &= T_1[U_0(\psi(z))F\circ\psi(z)] \\ &= U_0(\psi\circ\varphi(z))F\circ\psi\circ\varphi(z)\cdot\varphi'(z) \\ &= U(z)F(z) \end{aligned}$$

where $U(z) = U_0(z)\cdot\varphi'(z)$. Thus, if we can show that $U(\cdot)$ is a constant isometry of $\mathscr{H}$, we would obtain

$$TF(z) = T_1^{-1}UF(z) = UF(\psi(z))\cdot\psi'(z)$$

thus completing the proof.

Thus suppose that, with respect to our orthonormal basis $\{e_1, \dots, e_n\}$, for $z \in D$ the matrix of $U(z)$ is

$$\begin{pmatrix} u_{11}(z) & \dots & u_{1n}(z) \\ \vdots & & \vdots \\ u_{n1}(z) & \dots & u_{nn}(z) \end{pmatrix}.$$

Arguing as in the proof of Lemma 2 we see that each of the entries $u_{ij}(\cdot)$ are H^1 functions, and we use the same symbol u_{ij} to denote the corresponding function on ∂D.

If $f \in H^1$ and $1 \le j \le n$, consider the element $fe_j \in H^1_{\mathscr{H}}$. We have

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi}|f|\,dt = \|f\|_1 = \|T_1T(fe_j)\|_1 &= \left\|\sum_{i=1}^n f\cdot u_{ij}e_i\right\|_1 \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f|\left(\sum_{i=1}^n |u_{ij}|^2\right)^{1/2} dt. \end{aligned}$$

Hence,

$$0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f|\left[\left(\sum_{i=1}^n |u_{ij}|^2\right)^{1/2} - 1\right] dt,$$

and, since the moduli of H^1 functions are dense in the set of nonnegative, real-valued elements of $L^1(\partial D)$, we conclude that for each j, $1 \le j \le n$,

$$\sum_{k=1}^{n} |u_{kj}(e^{it})|^2 = 1, \qquad \text{a.e. on } \partial D, \tag{6}$$

which is to say that the column vectors of $U(e^{it})$ are unitary vectors on ∂D for almost all e^{it}. Note that, as a consequence, the entries $u_{kj}(\cdot)$ are not only H^1 elements but, in fact, H^∞ elements.

Since $\sum_{k=1}^{n} |u_{kj}(\cdot)|^2$ is subharmonic on D, we have

$$\sum_{k=1}^{n} |u_{kj}(z)|^2 \leq 1, \qquad z \in D. \tag{7}$$

Hence, for $z \in D$, the column vectors of $U(z)$ have length not greater than 1. Also if $1 \leq j < m \leq n$ consider, for $f \in H^1$, the element $fe_j + fe_m$ of $H^1_{\mathscr{H}}$. We have

$$\begin{aligned}\sqrt{2} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|\, dt = \|fe_j + fe_m\|_1 &= \|T_1 T(fe_j + fe_m)\|_1 \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f| \left(\sum_{k=1}^{n} |u_{kj} + u_{km}|^2 \right)^{1/2} dt.\end{aligned}$$

Thus

$$0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f| \left[\left(\sum_{k=1}^{n} |u_{kj} + u_{km}|^2 \right)^{1/2} - \sqrt{2} \right] dt,$$

so again using the density of the moduli of H^1 elements in the set of nonnegative, real-valued elements of $L^1(\partial D)$ we get that $\sum_{k=1}^{n} |u_{kj} + u_{km}|^2 = 2$ a.e. on ∂D. Thus

$$\sum_{k=1}^{n} [|u_{kj}(e^{it})|^2 + |u_{km}(e^{it})|^2 + 2\,\mathrm{Re}(u_{kj}(e^{it})\bar{u}_{km}(e^{it}))] = 2$$

which, together with (6) gives

$$\sum_{k=1}^{n} \mathrm{Re}(u_{kj}(e^{it})\bar{u}_{km}(e^{it})) = 0 \qquad \text{a.e. on } \partial D. \tag{8}$$

If we replace $fe_j + fe_m$ by $fe_j + ife_m$, the same argument then gives

$$\sum_{k=1}^{n} \mathrm{Im}(u_{kj}(e^{it})\bar{u}_{km}(e^{it})) = 0 \qquad \text{a.e. on } \partial D \tag{9}$$

so that (8) and (9) together give

$$\sum_{k=1}^{n} u_{kj}(e^{it})\bar{u}_{km}(e^{it}) = 0 \qquad \text{a.e. on } \partial D.$$

We have thus shown that $U(\cdot)$ is made up of column vectors which are of unit length and pairwise orthogonal a.e. on ∂D. That is, $U(e^{it})$ is a.e. a unitary operator on ∂D, and if we denote by $V(e^{it}) = [v_{kj}(e^{it})]$ the operator $U^*(e^{it})$, then the rule for computing the inverse of a matrix shows that the entires $v_{kj}(e^{it})$, considered as functions on ∂D, belong to H^∞ and we use the same symbols $v_{kj}(\cdot)$ to denote the corresponding functions defined on the disc.

An argument analogous to that which produced (7) shows that for each j, $1 \le j \le n$, and each $z \in D$,

$$\sum_{k=1}^{n} |v_{kj}(z)|^2 \le 1$$

and since the rows, as well as the columns of $[v_{kj}(e^{it})]$ have unit length a.e. we have, for $1 \le j \le n$,

$$\sum_{k=1}^{n} |v_{jk}(z)|^2 \le 1, \qquad z \in D. \tag{10}$$

Thus

$$\sum_{k=1}^{n} v_{jk}(e^{it}) u_{km}(e^{it}) = \delta_{jm} \qquad \text{a.e. on } \partial D,$$

and hence

$$\sum_{k=1}^{n} v_{jk}(z) u_{km}(z) = \delta_{jm} \qquad \text{on all of } D. \tag{11}$$

And since $\sum_{k=1}^{n} v_{jk}(z) u_{km}(z)$ is equal to the inner product

$$\langle v_{j1}(z)e_1 + \cdots + v_{jn}(z)e_n,\ \bar{u}_{1j}(z)e_1 + \cdots + \bar{u}_{nj}(z)e_n \rangle,$$

(11), together with (7) and (10), shows that $\sum_{k=1}^{n} |u_{kj}(z)|^2 = 1$ for all $z \in D$.

We have thus shown that, for $1 \le j \le n$, $z \to U(z)e_j$ is a vector-valued function defined on D to $\mathscr{H}$ such that $\|U(z)e_j\| = 1$ for $z \in D$. Hence the strong maximum modulus theorem for analytic vector-valued functions [12, Theorem 3.2] implies that $U(z)e_j$ is constant. Hence $U(\cdot)$ is a constant and the proof is complete.

3. Remarks and problems

(a) Can one establish the theorem of this article for H^1_E, where E belongs to a class of finite-dimensional Banach spaces properly containing Hilbert space? In [3] necessary and sufficient conditions were obtained on a finite-dimensional complex Banach space E which allow the description of the isometries of $H^\infty_{\mathscr{H}}$ given in [1] to be extended to H^∞_E. Thus, can one similarly characterize those finite-dimensional Banach spaces E which are such that the H^1 theorem established in this article extends to H^1_E? Since the condition on E obtained in the

H^{∞} case was that it not admit nontrivial L^{∞}-summands, it might be tempting to conjecture that the proper H^1 condition is the absence of L^1-summands. This condition is clearly necessary, but is it sufficient?

(b) Quite recently Lin [9] has been able to extend the H^{∞} result of [1] to H_E^{∞}, for certain infinite-dimensional Banach spaces E. Thus does the H^1 theorem of our paper have an analogue for infinite-dimensional range spaces? Quite obviously, many of the arguments used in this article are finite-dimensional in nature.

(c) The isometries of H^p for $1 < p < \infty$, $p \neq 2$, have been described by Forelli [6]. To the best of the authors' knowledge, there is no formulation of Forelli's theorem for vector-valued functions which exists in the literature.

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