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HYPERTRANSCENDENCE OF THE FUNCTIONAL EQUATION $g(x^2) = [g(x)]^2 + cx$

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ABSTRACT. The functional equation $g(x^2) = [g(x)]^2 + cx$ has a unique non-trivial solution that is analytic at zero. We show, for $c > 0$, that the solution of this equation satisfies no algebraic differential equation.

1. INTRODUCTION

The functional equation

$$g_1(x^2) = [g_1(x)]^2 + cx \tag{1.1}$$

has a unique non-trivial analytic solution in a neighbourhood of zero. Hence, on setting $g(x) := g_1(x^2)/x$, the functional equation

$$g(x^2) = [g(x)]^2 + c \tag{1.2}$$

has a unique meromorphic solution with a simple pole at zero. This functional equation was studied by Wedderburn [11], who observed that the solution to (1.1) when $c = 2$ arises as a generating function in a bracketing problem. Specifically the nth coefficient of g_1 counts the number of commutative non-associative bracketings of n objects. The more familiar problem of bracketing n objects in a nonassociative, noncommutative fashion, or equivalently, of making sense of an n-fold exponential

$$x^{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}$$

gives rise to a simpler functional equation, namely

$$f(x) = [f(x)]^2 + x$$

which has the algebraic solution

$$f(x) = \frac{1 - \sqrt{1 - 4x}}{2} = \sum_{n=1}^{\infty} \frac{(2n-2)!}{(n-1)!n!} x^n$$

and gives rise to the Catalan numbers.

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Wedderburn [11] analyses the analytic nature of the solutions of (1.1). Much more recently and entirely independently Mahler [8] also treated in detail the analytic nature of the solutions of (1.1).

The main content of this paper is contained in Theorem 1. Theorem 1 shows that the solutions of (1.1) or (1.2), for any positive c, satisfy no algebraic differential equation of any order (see §2). An algebraic differential equation is an equation

$$\Omega := \Omega(x, f(x), f'(x), \dots, f^n(x)) = 0$$

where Ω is a polynomial in $n+2$ variables. A function that satisfies no such equation is called *hypertranscendental* or *transcendentally transcendental*. Most special functions, precisely because they arise as solutions of such equations, are not hypertranscendental. An exception is the gamma function which was proved hypertranscendental by Hölder. For more on hypertranscendence see [5], [6] and particularly [10].

The functional equation (1.2) also arises in an entirely different way. Consider the two-term iteration

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad a_0 := 1 \tag{1.3}$$

and

$$b_{n+1} = \frac{a_n^2 + b_n^2}{a_n + b_n}, \qquad b_0 := x. \tag{1.4}$$

this iteration converges quadratically in a neighbourhood of 1 to an analytic function. The iteration and its various relatives were studied by Lehmer in [4] (see also [2]).

It had arisen previously in a letter of Stieltjes to Hermite in 1891. Stieltjes had constructed a formal power series L that uniformized the iteration in the sense that

$$L(q^2) = \frac{L(q) + L(-q)}{2} \tag{1.5}$$

and

$$L(-q^2) = \frac{L^2(q) + L^2(-q)}{L(q) + L(-q)}. \tag{1.6}$$

The normalization $L(0) = 1$ ensures that L is unique. Stieltjes apparently thought that L had zero radius of convergence, though in fact the radius of convergence is .634.... If we set

$$G(q^2) := \frac{4}{\frac{L(-q)}{L(q)} - 1} + 2,$$

then G has a simple pole at zero and, from (1.5) and (1.6),

$$G(q^2) = G(q)^2 + 2.$$

Thus G satisfies (1.2) with $c = 2$. In particular G is hypertranscendental and so is L. (See [1].)

This is in sharp contrast to the more familiar quadratic iteration, the AGM. The arithmetic-geometric mean iteration of Gauss and Legendre is

$$(1.7) \qquad a_{n+1} = \frac{a_n + b_n}{2}, \qquad a_0 = 1,$$

$$(1.8) \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad b_0 = x.$$

It is uniformized by the square of the theta function $\theta_3(q)$. If

$$T(q) := [\theta_3(q)]^2 = \left[\sum_{n=-\infty}^{\infty} q^{n^2}\right]^2$$

then

$$(1.9) \qquad T(q^2) = \frac{T(q) + T(-q)}{2}$$

and

$$(1.10) \qquad T(-q^2) = \sqrt{T(q)T(-q)}.$$

The common limit of (1.7) and (1.8) is an elliptic integral and solves a second order linear algebraic d.e. The theta function θ_3 is a modular form and satisfies a nonlinear algebraic d.e. and thus is not hypertranscendental. For more on these matters see [1], [2], [3] and [10].

Much of the point of this paper is to show that Lehmer's and Stieltjes's iteration belongs to a different analytic preserve than the arithmetic-geometric mean iteration.

2. Hypertranscendence of Wedderburn's Function

Lemma 1. *The functional equation, with $m > 1$ and $n_1 \geq n_2 > 0$,*

$$x^n \cdot R(x^m + c) = Ax^n \cdot R(x) + B$$

has no nonconstant rational solution if $c > 0$.

Proof. Suppose the above equation has a solution of the form $p(x)/q(x)$ where p and q are polynomials (with no common roots). A consideration of the degrees shows that $q(x)$ cannot be constant. Thus comparing poles on both sides shows that for some $h > 0$

$$q(x^m + c) = x^h(q^*(x)) \qquad \text{where } q^*(x) | q(x).$$

Now if $\alpha \geq 0$ is a root of $q(x^m + c)$ then $\alpha^m + c$ is a root of $x^h q^*(x)$ and hence $\alpha^m + c$ also is a root of $q(x^m + c)$. We know that 0 is a root of $q(x^m + c)$

because it is a root of $x^h(q^*(x))$, and we have

$$\begin{aligned} \ell_0 &:= 0, \\ \ell_1 &:= 0^m + c, \\ \ell_2 &:= c^m + c, \\ &\vdots \\ \ell_{n+1} &:= (\ell_n)^m + c \end{aligned}$$

are all roots of q. This however is an increasing sequence which generates infinitely many different roots of the polynomial q. This is impossible. □

Theorem 1. *Suppose F is analytic on $[\delta, \infty)$ and F satisfies a functional equation*

$$\alpha F(x) = F(x^2 + c)$$

where $\alpha \neq 1$ and $c > 0$; then F satisfies no algebraic differential equation.

Proof. Suppose F satisfies an algebraic differential equation $\Omega := \Omega(x)$. A monomial in Ω is a term

$$M(x) := r(x)[F(x)]^{m_0}[F^{(1)}(x)]^{m_1} \cdots [F^{(n)}(x)]^{m_n} \tag{2.1}$$

where r is a rational function in x. The degree of the monomial is

$$\mathrm{DEG}(M) := \sum_{i=1}^{n} i \cdot m_i \tag{2.2}$$

and the order of the monomial is

$$\mathrm{ORD}(M) = \sum_{i=0}^{n} m_i . \tag{2.3}$$

Observe that

$$\begin{aligned} \alpha F'(x) &= 2xF'(x^2 + c), \\ \alpha F''(x) &= 4x^2 F''(x^2 + c) + 2F'(x^2 + c), \\ &\vdots \\ \alpha F^{(n)}(x) &= 2^n x^n F^{(n)}(x^2 + c) \\ &\quad + A x^{n-2} F^{(n-1)}(x^2 + c) \qquad (A > 0) \\ &\quad \vdots \\ &\quad + \text{lower derivative terms}. \end{aligned} \tag{2.4}$$

Thus

$$\begin{aligned} M(x) &= r(x)[F(x)]^{m_0}[F^{(1)}(x)]^{m_1} \cdots [F^{(n)}(x)]^{m_n} \\ &= r(x) \frac{2^{\mathrm{DEG}(M)} x^{\mathrm{DEG}(M)}}{\alpha^{\mathrm{ORD}(M)}} [F(x^2 + c)]^{m_0} \cdots [F^{(n)}(x^2 + c)]^{m_n} \\ &\quad + \text{lower degree terms}. \end{aligned} \tag{2.5}$$

Now suppose Ω is minimal in the sense that the degree of the maximal monomials in Ω is minimal, and among those of the same degree pick a Ω with fewest maximal terms. Let M^* be such a maximal degree monomial and suppose that the Ω is normalized so that the r associated with M^* is 1. Let $\overline{\Omega}$ be the algebraic differential equation obtained from Ω by replacing each $F^{(i)}(x)$ by the appropriate combinations of $F^{(i)}(x^2+c)$ as in (4). Observe by (5) that $\overline{\Omega}$ has the same maximal degree terms as $\Omega(x^2+c)$ (i.e. Ω with x replaced by x^2+c). It follows that, monomial by monomial,

$$(2.6)\qquad \frac{2^{\mathrm{DEG}(M^*)}x^{\mathrm{DEG}(M^*)}}{\alpha^{\mathrm{ORD}(M^*)}}\Omega(x^2+c) = \overline{\Omega}(x)$$

for otherwise the difference would be an algebraic differential equation for F with fewer maximal terms (we would subtract out M^*). If N is any other monomial of maximal degree with rational coefficient r_N then, from (6),

$$\frac{2^{\mathrm{DEG}(M^*)}x^{\mathrm{DEG}(M^*)}}{\alpha^{\mathrm{ORD}(M^*)}}r_N(x^2+c) = r_N(x)\frac{2^{\mathrm{DEG}(N)}x^{\mathrm{DEG}(N)}}{\alpha^{\mathrm{ORD}(N)}}$$

(since N is maximal no other terms come into play). From the Lemma we see that r_N is constant and that $\mathrm{ORD}(N) = \mathrm{ORD}(M^*)$. Suppose that $m_0, \dots, m_n$ are the exponents of $F^{(0)}, \dots, F^{(n)}$ in some maximal monomial M. Suppose out of all such maximal monomials M has the largest possible value of m_n and then the largest possible value of m_{n-1} etc. Let i be the least index for which no other maximal monomial has all strictly higher indexes agreeing with M. Consider the monomial W with exponents $m_0, \dots, m_{i-1}+1, m_i-1, \dots, m_n$. Such a monomial arises in (5) by picking the highest derivative (as in (4)) associated with all the exponents except i and $i-1$. One picks the second highest derivative in one of the terms associated with i in the substitution (5). Then, for some constant $D \neq 0$,

$$(2.7)\qquad \begin{aligned} M(x) = {} & \frac{2^{\mathrm{DEG}(M)}x^{\mathrm{DEG}(M)}}{\alpha^{\mathrm{ORD}(M)}}[F(x^2+c)]^{m_0}\cdots[F^{(n)}(x^2+c)]^{m_n} \\ & + Dx^{\mathrm{DEG}(M)-2}[F(x^2+c)]^{m_0}\cdots[F^{(i-1)}(x^2+c)]^{m_{i-1}+1} \\ & \cdot[F^{(i)}(x^2+c)]^{m_i-1}\cdots[F^{(n)}(x^2+c)]^{m_n} \\ & + \cdots . \end{aligned}$$

Assume initially that $i \geq 2$ so that this construction works (if $m_i = 0$ one uses the first exponent smaller than m_i which isn't zero*). Note also, unless the lower order monomial is formed by a process as above (i.e. picking exactly one nonmaximal derivative in (4)), that the degree of the resulting piece is at least two less than the degree of M. Note that

$$\mathrm{ORD}(W) = \mathrm{ORD}(M)$$

* Added in proof. If no such exponent exists some additional argument is required.

and that

$$\mathrm{DEG}(W) = \mathrm{DEG}(M) - 1 .$$

In particular if any other monomial contributes (as in (7)) to the formation of W it must be of the form

$$[F^{(n)}(x)]^{m_n} \cdots [F^{(i+1)}(x)]^{m_{i+1}} \cdots ,$$

which contradicts the maximality assumptions on M. It follows that W arises only as in (7) and in order that W vanish in (6) it must hold that

$$x^{\mathrm{DEG}(M)} r_W(x^2 + c) = E x^{\mathrm{DEG}(W)} r_W(x) + D x^{\mathrm{DEG}(M)-2}$$

where $D \neq 0$. This is impossible by the Lemma.

It remains to dispose of the cases where $i = 0$ and $i = 1$.

For $i = 1$, we can use the same argument only now we consider the monomial that arises on reducing the $F^{(2)}$ term, provided such a term exists, by 1 and increasing the $F^{(1)}$ term. The details are essentially the same. The $i = 0$ case can't occur unless there is a unique monomial of maximal degree. This follows because such monomials of maximal degree have the same maximal order. If there is a unique monomial of maximal degree then reducing any term works, provided the term has index at least two. Since everything we have said applied if we took our maximums over monomials containing at least 2nd derivatives, we can reduce to the case that $F^{(1)}$ actually satisfies an algebraic equation in F. That is

$$\Omega(x) = \sum_{h,j} r_{h,j}(x)(F^{(1)})^h (F)^j .$$

Given that we normalize the maximal term as before, we see from (4) and (6) that

$$\frac{2^{\mathrm{DEG}(M)} x^{\mathrm{DEG}(M)}}{\alpha^{\mathrm{ORD}(M)}} r_{h,j}(x^2 + c) = (2x)^h r_{h,j}(x)$$

which also violates the Lemma. (Unless $h = \mathrm{DEG}(M)$ which gives the non-meromorphic solution, with $r_{k,j}$ constant, of $F = \log x$). □

Remarks. (1) Essentially the same argument applies to show that solutions of

$$\alpha F(x) = F(x^n + c), \qquad c > 0,\ \alpha \neq 1,$$

are hypertranscendental for $n > 2$.

(2) Wedderburn's function satisfies

$$g(x^2) = [g(x)]^2 + 2$$

and hence $G := g^{-1}$ satisfies

$$G(x)^2 = G(x^2 + 2).$$

Thus $F = \log G$ satisfies

$$2F(x) = F(x^2 + 2).$$

Furthermore g is decreasing and analytic on an interval $[0, \beta]$ with $g(0) = \infty$. Thus G is well defined and analytic on some interval $[\delta, \infty)$ and hence so is F. Thus, as a corollary to the theorem we see that g is hypertranscendental.

(3) If c is algebraic then the function g of (1.2) maps algebraic numbers to transcendentals. This follows from results of Mahler [6], [7].

(4) The method of proof in Theorem 1 is not unlike the method employed by Hölder to prove hypertranscendence of the gamma function last century. Indeed Ritt's treatment of the hypertranscendence of functional equations of the form

$$f(2x) = R(f(x))$$

where R is rational also uses similar techniques [9]. The equation

$$g(x^2) = [g(x)]^2 + c$$

transforms to the equation

$$g^*(2y) = [g^*(y)]^2 + c$$

under the transformation $g^*(y) = g(e^x)$. However to apply Ritt's results we would have to make various assumptions on the meromorphic nature of g^* that we can't reasonably make.

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