PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 107, Number 1, September 1989

OPERATOR RANGES AND COMPLETELY BOUNDED HOMOMORPHISMS

D. BENJAMIN MATHES

(Communicated by John B. Conway)

ABSTRACT. In this paper it is shown that the set of invariant operator ranges that induce completely bounded homomorphisms is a sublattice of $\mathrm{Lat}_{1/2}\mathfrak{A}$, when $\mathfrak{A}$ is any norm closed algebra of operators on a Hilbert space. A characterization of this sublattice is given, and several concrete examples are discussed.

1. INTRODUCTION

The most famous question in the theory of operator ranges is Dixmier's. If $\mathbf{H}$ is a Hilbert space, $\mathfrak{A}$ is a unital C^*-algebra of operators on $\mathbf{H}$, and T is an operator on $\mathbf{H}$ whose range is invariant under every operator in $\mathfrak{A}$, Dixmier's question asks if there exists an operator B in the commutant of $\mathfrak{A}$ with the same range as that of T. An affirmative answer is known to be true for a large class of C^*-algebras (see [6], [18]), but the general case is still unresolved. In [6] Foiaş proves that given an operator T whose range is invariant under an algebra of operators $\mathfrak{A}$, there exists a homomorphism Φ_T that maps $\mathfrak{A}$ into $B(\mathbf{H})$ (the algebra of operators on $\mathbf{H}$), and satisfies $AT = T\Phi_T(A)$ for all $A \in \mathfrak{A}$. Furthermore, this homomorphism is bounded whenever $\mathfrak{A}$ is norm closed. In case $\mathfrak{A}$ is a C^*-algebra, Foiaş proves in the same paper that if Φ_T is similar to a $*$-homomorphism, then there exists an operator $B \in \mathfrak{A}'$ (the commutant of $\mathfrak{A}$) with the same range as that of T. In [7] Haagerup proves that a homomorphism of a unital C^*-algebra into $B(\mathbf{H})$ is similar to a $*$-homomorphism if and only if it is completely bounded. These two results together say that Dixmier's question has an affirmative answer if and only if invariant ranges of unital C^*-algebras always induce completely bounded homomorphisms. If $\mathrm{Lat}_{1/2}\mathfrak{A}$ denotes the lattice of invariant operator ranges of $\mathfrak{A}$ (with intersection and linear span as lattice operations), we see from the comments above that

$$\{B\mathbf{H} | B \in \mathfrak{A}'\} = \{T\mathbf{H} \in \mathrm{Lat}_{1/2}\mathfrak{A} | \Phi_T \text{ completely bounded}\}.$$

Received by the editors October 6, 1988. Portions of this paper were presented on May 22, 1989 to the Canadian Operator Symposium at the University of Toronto.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47C05; Secondary 47A05.

Key words and phrases. Invariant operator range, completely bounded homomorphism.

The results of this paper constitute a portion of the author's Ph. D. dissertation at the University of New Hampshire.

Suppose now that $\mathfrak{A}$ is the algebra of analytic Toeplitz operators. In [6] Foiaş proves that $\{B\mathbf{H}|B \in \mathfrak{A}\}$ is not a lattice (it fails to be closed under linear span); thus the analog of Dixmier's question (is every element of $\operatorname{Lat}_{1/2}\mathfrak{A}$ the range of some operator in $\mathfrak{A}' = \mathfrak{A}$) is easily answered in the negative. In [17] Roy proves that for this algebra

$$\mathfrak{L}_\infty \equiv \{T\mathbf{H}^{(\infty)}|T \in B(\mathbf{H}^{(\infty)}, \mathbf{H}); AT = TA^{(\infty)} \text{ for all } A \in \mathfrak{A}\}$$

is a sublattice of $\operatorname{Lat}_{1/2}\mathfrak{A}$ and conjectures that it is all of $\operatorname{Lat}_{1/2}\mathfrak{A}$. Combining results from [11] and [14] it is immediately deduced that

$$\mathfrak{L}_\infty = \{T\mathbf{H} \in \operatorname{Lat}_{1/2}\mathfrak{A}|\Phi_T \text{ completely bounded}\}.$$

Thus, Roy's conjecture is that every element of $\operatorname{Lat}_{1/2}\mathfrak{A}$ induces a completely bounded homomorphism, when $\mathfrak{A}$ is the algebra of analytic Toeplitz operators.

In this paper we prove that if $\mathfrak{A}$ is an arbitrary unital norm closed algebra of operators on a Hilbert space $\mathbf{H}$, then the subset of $\operatorname{Lat}_{1/2}\mathfrak{A}$ consisting of the invariant ranges that induce completely bounded homomorphisms, which we denote $\operatorname{Lat}_{cb}\mathfrak{A}$, forms a sublattice of $\operatorname{Lat}_{1/2}\mathfrak{A}$. We then associate a norm closed algebra $\mathfrak{B}_{\mathfrak{A}}$ with $\mathfrak{A}$ and prove that $\operatorname{Lat}_{1/2}\mathfrak{B}_{\mathfrak{A}}$ is always lattice isomorphic to $\operatorname{Lat}_{cb}\mathfrak{A}$, under an isomorphism that restricts to an isomorphism of $\operatorname{Lat}\mathfrak{B}_{\mathfrak{A}}$ with $\operatorname{Lat}\mathfrak{A}$. We use this isomorphism to carry results about $\operatorname{Lat}_{1/2}\mathfrak{A}$ over to results about $\operatorname{Lat}_{cb}\mathfrak{A}$. Finally, we obtain a general representation of $\operatorname{Lat}_{cb}\mathfrak{A}$ as the set of ranges of operators T that satisfy $AT = T\Phi(A)$ for all $A \in \mathfrak{A}$, where Φ is a complete contraction with domain $\mathfrak{A}$.

2. Invariant ranges inducing completely bounded homomorphisms

Let $\mathfrak{A}$ be a unital norm closed algebra of operators acting on a Hilbert space $\mathbf{H}$. A bounded linear map $\Phi\colon \mathfrak{A} \to B(\mathbf{H})$ is called completely bounded provided $\|\Phi\|_{cb} \equiv \sup\{\|\Phi \otimes 1_n\| | n = 1, 2, \ldots\}$ is finite, where 1_n is the identity map on $B(\mathbb{C}^n)$ and $\Phi \otimes 1_n$ acts on an elementary tensor $A \otimes M$ in $\mathfrak{A} \otimes B(\mathbb{C}^n)$ by $(\Phi \otimes 1_n)(A \otimes M) = \Phi(A) \otimes M$.

Suppose $T \in B(\mathbf{K}, \mathbf{H})$, $S \in B(\mathbf{M}, \mathbf{H})$ and $T\mathbf{H} = S\mathbf{H} \in \operatorname{Lat}_{1/2}\mathfrak{A}$. If T_0 is the restriction of T to $(\ker T)^\perp$, then $\Phi_T(A) = T_0^{-1}AT$ for every $A \in \mathfrak{A}$ (see [6]). We assert that Φ_T is completely bounded if and only if Φ_S is completely bounded. Since $T_0\mathbf{H} = T\mathbf{H}$, we have Φ_{T_0} mapping $\mathfrak{A}$ into $B((\ker T)^\perp)$. The operator $T_0^{-1}T$ is the orthogonal projection of $\mathbf{K}$ onto $(\ker T)^\perp$, and

$$\Phi_{T_0}(A) = (T_0^{-1}T)\Phi_T(A)|_{\ker T^\perp},$$

thus Φ_{T_0} is completely bounded whenever Φ_T is, since a compression of a completely bounded map is completely bounded. On the other hand, note that $\Phi_T = \Phi_{T_0} \oplus 0$ relative to the decomposition of $\mathbf{K}$ as $(\ker T)^\perp \oplus \ker T$, so Φ_T is completely bounded whenever Φ_{T_0} is. We have that $\Phi_{T_0}(\Phi_{S_0})$ is completely bounded if and only if $\Phi_T(\Phi_S)$ is completely bounded. To prove the assertion, we need only prove that Φ_{S_0} is completely bounded if and only if Φ_{T_0} is; but $\Phi_{S_0}(A) = R^{-1}\Phi_{T_0}(A)R$, where R is the invertible operator $R = S_0^{-1}T_0$ in $B((\ker T)^{\perp}, (\ker S)^{\perp})$. The set $\mathrm{Lat}_{1/2}\mathfrak{A}$ is thus partitioned into two pieces, those invariant ranges that induce completely bounded homomorphisms, and those that do not. We call the set of invariant ranges that induce completely bounded homomorphisms $\mathrm{Lat}_{\mathrm{cb}}\mathfrak{A}$ and proceed to justify the name.

Theorem 2.1. *If* $\mathfrak{A}$ *is a norm closed unital subalgebra of* $B(\mathbf{H})$, *then* $\mathrm{Lat}_{\mathrm{cb}}\mathfrak{A}$ *is a sublattice of* $\mathrm{Lat}_{1/2}\mathfrak{A}$.

Proof. Assume S, $T \in B(\mathbf{H})$ and $S\mathbf{H}$, $T\mathbf{H} \in \mathrm{Lat}_{\mathrm{cb}}\mathfrak{A}$, hence Φ_S, Φ_T, and $\Phi_S \oplus \Phi_T$ are completely bounded maps. Define $W\colon \mathbf{H}^{(2)} \to \mathbf{H}$ by $W(u, v) = Su + Tv$, thus $W\mathbf{H}^{(2)} = S\mathbf{H} + T\mathbf{H}$, and

$$\begin{aligned}\Phi_W(A)(u,v) &= W_0^{-1}AW(u,v) = W_0^{-1}(ASu + ATv)\\ &= W_0^{-1}(S\Phi_S(A) + T\Phi_T(A)) = W_0^{-1}W(\Phi_S \oplus \Phi_T)(A)(u,v).\end{aligned}$$

Since $W_0^{-1}W$ is the orthogonal projection of $\mathbf{H}^{(2)}$ onto $\ker W^{\perp}$, we see that Φ_W is a compression of the completely bounded map $\Phi_S \oplus \Phi_T$. Hence Φ_W is completely bounded, and $S\mathbf{H} + T\mathbf{H} \in \mathrm{Lat}_{\mathrm{cb}}\mathfrak{A}$.

We now show that $S\mathbf{H} \cap T\mathbf{H} \in \mathrm{Lat}_{\mathrm{cb}}\mathfrak{A}$. Define $Y\colon \mathbf{H}^{(2)} \to \mathbf{H}^{(2)}$ by the matrix

$$Y = \begin{pmatrix} S & -T \\ 0 & 0 \end{pmatrix}.$$

Let $\mathbf{K} = \ker Y$, define $P\colon \mathbf{H}^{(2)} \to \mathbf{H}$ by $P(u, v) = u$, and let $W = SP|_{\mathbf{K}}$. Notice that $(u, v) \in \mathbf{K}$ if and only if $Su = Tv$, and that $W\mathbf{K} = S\mathbf{H} \cap T\mathbf{H}$.

Suppose $(u, v) \in \mathbf{K}$. We assert that for every $A \in \mathfrak{A}$, $(\Phi_S \oplus \Phi_T)(A)(u, v) \in \mathbf{K}$. Note that $(\Phi_S \oplus \Phi_T)(A)(u, v) \in \mathbf{K}$ if and only if $S\Phi_S(A)u = T\Phi_T(A)v$. Since $Su = Tv$, $S\Phi_S(A)u = ASu = ATv = T\Phi_T(A)v$, and the assertion follows.

We now look at Φ_W:

$$\begin{aligned}\Phi_W(A)(u,v) &= W_0^{-1}AW(u,v) = W_0^{-1}ASu = W_0^{-1}S(\Phi_S(A)u)\\ &= W_0^{-1}W(\Phi_S(A)u, \Phi_T(A)v) = W_0^{-1}W(\Phi_S \oplus \Phi_T)(A)(u,v).\end{aligned}$$

Once again, Φ_W is a compression of $\Phi_S \oplus \Phi_T$. Thus Φ_W is completely bounded and $S\mathbf{H} \cap T\mathbf{H} \in \mathrm{Lat}_{\mathrm{cb}}\mathfrak{A}$. $\square$

Paulsen [14] has shown that if $\mathfrak{A}$ is a C^*-algebra of operators, then the range of an operator T is in $\mathrm{Lat}_{\mathrm{cb}}\mathfrak{A}$ if and only if the range of $T \otimes 1$ is in $\mathrm{Lat}_{1/2}\mathfrak{A} \otimes \mathfrak{K}$, where $\mathfrak{K}$ denotes the compact operators on an infinite dimensional separable Hilbert space $\mathbf{K}$, and 1 is the identity map on $\mathbf{K}$. In his paper, Paulsen only concerns himself with the case when $\mathfrak{A}$ is a C^*-algebra of operators, but his result is true whenever $\mathfrak{A}$ is an arbitrary norm closed algebra of operators, when the tensor product $\mathfrak{A} \otimes \mathfrak{K}$ is taken to mean the norm closed algebra generated by $\{A \otimes K | A \in \mathfrak{A}; K \in \mathfrak{K}\}$ in $B(\mathbf{H} \otimes \mathbf{K})$.

Theorem 2.2. *Assume* $\mathfrak{A}$ *is a norm closed unital subalgebra of* $B(\mathbf{H})$. *Then* $\mathrm{Lat}_{\mathrm{cb}}\,\mathfrak{A}$ *is lattice isomorphic to* $\mathrm{Lat}_{1/2}\,\mathfrak{A}\otimes\mathfrak{K}$, *and the lattice isomorphism may be chosen so that the restriction to* $\mathrm{Lat}\,\mathfrak{A}$ *is a lattice isomorphism of* $\mathrm{Lat}\,\mathfrak{A}$ *onto* $\mathrm{Lat}\,\mathfrak{A}\otimes\mathfrak{K}$.

Proof. Suppose T, $S\in B(\mathbf{H})$ and $T\mathbf{H}=S\mathbf{H}$. Then it is clear that

$$(T\otimes 1)\mathbf{H}\otimes\mathbf{K}=(S\otimes 1)\mathbf{H}\otimes\mathbf{K}.$$

Thus if $\mathfrak{R}$ is any operator range in $\mathbf{H}$ we may define (unambiguously) $\mathfrak{R}\otimes 1$ to be the operator range in $\mathbf{H}\otimes\mathbf{K}$ that satisfies $T\mathbf{H}=\mathfrak{R}$ if and only if $(T\otimes 1)\mathbf{H}\otimes\mathbf{K}=\mathfrak{R}\otimes 1$. It follows that $\mathfrak{R}\to\mathfrak{R}\otimes 1$ is a well-defined map of the lattice of all operator ranges in $\mathbf{H}$ into the lattice of operator ranges in $\mathbf{H}\otimes\mathbf{K}$. It is immediately clear that the map is injective and that if preserves range inclusion.

To prove that the lattice operations are preserved, note that $C\mathbf{H}=S\mathbf{H}+T\mathbf{H}$ is equivalent to the range inclusion statement: $S\mathbf{H}$, $T\mathbf{H}\subset C\mathbf{H}$ and $S\mathbf{H}$, $T\mathbf{H}\subset B\mathbf{H}$ implies $C\mathbf{H}\subset B\mathbf{H}$. This fact, together with the fact that the set $\{\mathfrak{R}\otimes 1|\mathfrak{R}$ an operator range in $\mathbf{H}\}$ is a lattice (it is the set of ranges of operators from a von Neumann algebra [4], [5], [8]), implies linear span is preserved. Intersection is proved to be preserved in an analogous way. Therefore the map $\mathfrak{R}\to\mathfrak{R}\otimes 1$ is a lattice monomorphism of the lattice of all operator ranges in $\mathbf{H}$ into the lattice of all operator ranges in $\mathbf{H}\otimes\mathbf{K}$.

From Paulsen's result and what we have shown already, $\mathfrak{R}\to\mathfrak{R}\otimes 1$ is a lattice monomorphism of $\mathrm{Lat}_{\mathrm{cb}}\,\mathfrak{A}$ into $\mathrm{Lat}_{1/2}\,\mathfrak{A}\otimes\mathfrak{K}$, and it is trivial to verify that $\mathfrak{R}$ is closed if and only if $\mathfrak{R}\otimes 1$ is closed. Thus we are finished as soon as we prove that the mapping is onto $\mathrm{Lat}_{1/2}\,\mathfrak{A}\otimes\mathfrak{K}$.

The rest of the proof uses a technique that appears in the proof of Theorem 5 of [11]. Let $\mathfrak{D}\in\mathrm{Lat}_{1/2}\,\mathfrak{A}\otimes\mathfrak{K}$, and let $\mathscr{I}$ denote the scalar operators in $B(\mathbf{H})$. Since $\mathfrak{A}$ is unital, $\mathfrak{D}\in\mathrm{Lat}_{1/2}\mathscr{I}\otimes\mathfrak{K}$. By a result of Ong [12], $\mathfrak{D}\in\mathrm{Lat}_{1/2}(\mathscr{I}\otimes\mathfrak{K})^{-WOT}$. Since $(\mathscr{I}\otimes\mathfrak{K})^{-WOT}$ is a von Neumann algebra with Schwartz's property P, there exists $W\in(\mathscr{I}\otimes\mathfrak{K})'$ such that $W(\mathbf{H}\otimes\mathbf{K})=\mathfrak{D}$, by a result of Voiculescu [18]. Our result follows because

$$(\mathscr{I}\otimes\mathfrak{K})'=\{T\otimes 1|T\in B(\mathbf{H})\}.\quad\square$$

The following corollary follows immediately from a theorem of Azoff (see [1]), who identified a sublattice $\mathfrak{L}(\mathfrak{A})$ of $\mathrm{Lat}_{1/2}\,\mathfrak{A}$ (that he calls the characteristic manifolds of $\mathfrak{A}$) with the property that any unital subalgebra $\mathfrak{A}$ of $B(\mathbf{H})$ with $\mathfrak{L}(\mathfrak{A})=\{(0),\mathbf{H}\}$ must be weakly dense in $B(\mathbf{H})$. It is easily verified that when $\mathfrak{A}$ is norm closed, one has $\mathfrak{L}(\mathfrak{A})\subset\mathrm{Lat}_{\mathrm{cb}}\,\mathfrak{A}$, from which our corollary follows. We provide the following proof to illustrate how facts about $\mathrm{Lat}_{1/2}$ may be carried over to $\mathrm{Lat}_{\mathrm{cb}}$.

Corollary 2.3. *Assume* $\mathfrak{A}$ *is a weakly closed unital subalgebra of* $B(\mathbf{H})$. *If* $\mathrm{Lat}_{\mathrm{cb}}\,\mathfrak{A}=\{(0),\mathbf{H}\}$, *then* $\mathfrak{A}=B(\mathbf{H})$.

Proof. Assume $\mathfrak{A} \neq B(\mathbf{H})$. Then $(\mathfrak{A} \otimes \mathfrak{K})^{-WOT} \neq B(\mathbf{H} \otimes \mathbf{K})$, so by a theorem of Foiaş [6]

$$\operatorname{Lat}_{1/2}(\mathfrak{A} \otimes \mathfrak{K})^{-WOT} \neq \{(0), \mathbf{H} \otimes \mathbf{K}\}.$$

By Theorem 2.2 it follows that $\operatorname{Lat}_{\mathrm{cb}} \mathfrak{A} \neq \{(0), \mathbf{H}\}$. □

Ong proved that no invariant ranges of a norm closed algebra $\mathfrak{A}$ are lost when passing to the weak closure $\mathfrak{A}^{-WOT}$, provided the unit ball of $\mathfrak{A}$ is weakly dense in the ball of $\mathfrak{A}^{-WOT}$ (see [12]). The following theorem is the analogous statement for $\operatorname{Lat}_{\mathrm{cb}} \mathfrak{A}$.

Corollary 2.4. *Suppose $\mathfrak{A}$ is a norm closed unital subalgebra of $B(\mathbf{H})$ with the property that the unit ball of $\mathfrak{A}$ is weakly dense in the unit ball of $\mathfrak{A}^{-WOT}$. Then $\operatorname{Lat}_{\mathrm{cb}} \mathfrak{A} = \operatorname{Lat}_{\mathrm{cb}} \mathfrak{A}^{-WOT}$.*

Proof. It is clear that $\operatorname{Lat}_{\mathrm{cb}} \mathfrak{A}^{-WOT} \subset \operatorname{Lat}_{\mathrm{cb}} \mathfrak{A}$ (since the restriction of a completely bounded map on $\mathfrak{A}^{-WOT}$ to $\mathfrak{A}$ will be completely bounded), so assume that $\mathfrak{R} \in \operatorname{Lat}_{\mathrm{cb}} \mathfrak{A}$. We have that $\mathfrak{R} \otimes 1 \in \operatorname{Lat}_{1/2} \mathfrak{A} \otimes \mathfrak{K}$ by Paulsen's result [14], and thus $\mathfrak{R} \otimes 1 \in \operatorname{Lat}_{1/2}(\mathfrak{A} \otimes \mathfrak{K})^{-WOT}$ by a result of Ong [12]. Noting that $(\mathfrak{A} \otimes \mathfrak{K})^{-WOT} = (\mathfrak{A}^{-WOT} \otimes \mathfrak{K})^{-WOT}$ we have $\mathfrak{R} \otimes 1 \in \operatorname{Lat}_{1/2}(\mathfrak{A}^{-WOT} \otimes \mathfrak{K})^{-WOT}$ and $\mathfrak{R} \in \operatorname{Lat}_{\mathrm{cb}} \mathfrak{A}^{-WOT}$, again by Paulsen's result [14]. □

3. Algebras for which every invariant range induces a completely bounded homomorphism

It is clear from the comments in the introduction that if a C^*-algebra $\mathfrak{A}$ yields an affirmative answer to Dixmier's operator range question, then $\operatorname{Lat}_{1/2} \mathfrak{A} = \operatorname{Lat}_{cb} \mathfrak{A}$. In particular, if $\mathfrak{A}''$ has Schwartz's property P, then $\operatorname{Lat}_{1/2} \mathfrak{A} = \operatorname{Lat}_{\mathrm{cb}} \mathfrak{A}$. It seems that every norm closed algebra $\mathfrak{A}$ that has had its invariant operator range lattice classified satisfies $\operatorname{Lat}_{1/2} \mathfrak{A} = \operatorname{Lat}_{\mathrm{cb}} \mathfrak{A}$. We begin with another corollary to Theorem 2.2.

Corollary 3.1. *If $\mathfrak{A}$ is a norm closed unital subalgebra of $B(\mathbf{H})$, then*

$$\operatorname{Lat}_{\mathrm{cb}} \mathfrak{A} \otimes \mathfrak{K} = \operatorname{Lat}_{1/2} \mathfrak{A} \otimes \mathfrak{K}.$$

Proof. Assume that $\mathfrak{D} \in \operatorname{Lat}_{1/2} \mathfrak{A} \otimes \mathfrak{K}$. By Theorem 2.2 there exists $T \in B(\mathbf{H})$ such that $T\mathbf{H} \in \operatorname{Lat}_{\mathrm{cb}} \mathfrak{A}$ and $\mathfrak{D} = T\mathbf{H} \otimes 1 = (T \otimes 1)\mathbf{H} \otimes \mathbf{K}$. However,

$$\Phi_{T\otimes 1}(A \otimes K) = (T \otimes 1)_0^{-1}(A \otimes K)(T \otimes 1) = T_0^{-1} A T \otimes K = \Phi_T \otimes 1(A \otimes K),$$

where 1 is the identity map on $B(\mathbf{K})$. Finally, $\Phi_T \otimes 1$ is completely bounded since both Φ_T and 1 are (see Theorem 10.3 of [16]). □

Corollary 3.2. *There exists a von Neumann algebra $\mathfrak{B}$ without Schwartz's property P that satisfies $\operatorname{Lat}_{\mathrm{cb}} \mathfrak{B} = \operatorname{Lat}_{1/2} \mathfrak{B}$.*

Proof. Let $\mathfrak{A}$ be a factor in $B(\mathbf{H})$ without Schwartz's property P, and let $\mathfrak{B} = (\mathfrak{A} \otimes \mathfrak{K})^{-WOT}$. Then $\mathfrak{B}$ is a factor in $B(\mathbf{H} \otimes \mathbf{K})$ without Schwartz's property

P. (Prove that if $\mathfrak{B}$ is an approximately finite dimensional factor, then $\mathfrak{A}$ must be an approximately finite dimensional factor; a factor has property P if and only if it is approximately finite dimensional [2].) This corollary then follows from the previous corollary. □

If $0 \leq T \leq 1$ and $T = \int_{[0,1]} z\, dE(z)$, then it is proved in [10] that every operator that leaves $T\mathbf{H}^-$ invariant, and whose restriction to $T\mathbf{H}^-$ is upper triangular with respect to the decomposition $\bigoplus E((\frac{1}{2})^{i+1}, (\frac{1}{2})^i])\mathbf{H}$, leaves $T\mathbf{H}$ invariant. Thus, if a norm closed algebra $\mathfrak{A}$ has $T\mathbf{H}^- \in \operatorname{Lat}\mathfrak{A}$, and its restriction to $T\mathbf{H}^-$ is upper triangular with respect to this decomposition, then $\operatorname{ran} T \in \operatorname{Lat}_{1/2}\mathfrak{A}$. The following theorem asserts that invariant ranges of this kind always induce completely bounded homomorphisms.

Theorem 3.3. *Assume* $\mathfrak{A}$ *is a norm closed unital subalgebra of* $B(\mathbf{H})$, $\mathbf{M}_i \in \operatorname{Lat}\mathfrak{A}$ $(i = 0, 1, 2, \dots)$, $\mathbf{M}_0 \subset \mathbf{M}_1 \subset \mathbf{M}_2 \subset \cdots$, *and let* $\mathbf{N}_i = \mathbf{M}_i \ominus \mathbf{M}_{i-1}$, $(\mathbf{M}_{-1} = (0);\ i = 0, 1, 2, \dots)$. *Define* $T \equiv \operatorname{diag}(1, \frac{1}{2}, (\frac{1}{2})^2, (\frac{1}{2})^3, \dots)$ *relative to* $\oplus \mathbf{N}_i$ (*if* $x \in \mathbf{H}$, *write* $x = z + \sum n_i$ *with* $n_i \in \mathbf{N}_i$ *and* $z \in (\bigoplus \mathbf{N}_i)^\perp$; $Tx = \sum (\frac{1}{2})^i n_i$). *Then* $\operatorname{ran} T \in \operatorname{Lat}_{\mathrm{cb}}\mathfrak{A}$.

To prove this theorem, we introduce the concept of an infinite dimensional Schur product. Suppose $S \in B(\mathbf{H})$, $T \in B(\mathbf{l}^2)$, and $\lambda_{ij} = \langle Te_j, e_i\rangle$, where $\{e_i | i = 0, 1, \dots\}$ is the standard basis of $\mathbf{l}^2$. Assume $\mathfrak{F} = \{\mathbf{M}_i | i = 0, 1, \dots\}$ is a pairwise orthogonal family of subspaces that span $\mathbf{H}$ (we call $\mathfrak{F}$ total), and (S_{ij}) is the matrix of S relative to this family. Then we call the matrix $(\lambda_{ij} S_{ij})$ the *Schur product* of S and T relative to $\mathfrak{F}$ and denote it $\operatorname{Schur}_T(S)$. Schur products of finite matrices are discussed in [16], and the proof of the lemma below proceeds just as in the finite dimensional case.

Lemma 3.4. *If* $\mathfrak{F} = \{\mathbf{M}_i : i \in N\}$ *is a total orthogonal family of subspaces of* $\mathbf{H}$, $S \in B(\mathbf{H})$, *and* $T \in B(\mathbf{l}^2)$, *then* $\operatorname{Schur}_T(S)$ *is the matrix of an operator relative to* $\mathfrak{F}$ *and* $\operatorname{Schur}_T \colon B(\mathbf{H}) \to B(\mathbf{H})$ *is a completely bounded mapping with* $\|\operatorname{Schur}_T\|_{\mathrm{cb}} \leq \|T\|$.

Proof. Let $\Psi \colon B(\mathbf{H}) \to B(\mathbf{H} \otimes \mathbf{l}^2)$ be the mapping $\Psi(S) = S \otimes T = (1 \otimes T)(S \otimes 1)$. Then $\|\Psi\|_{\mathrm{cb}} \leq \|T\|$ (Ψ is a composition of the complete contraction $S \to S \otimes 1$ with the left multiplication operator on $B(\mathbf{H} \otimes \mathbf{l}^2)$ defined by $W \to (1 \otimes T)W$, which is completely bounded with cb-norm equal to $\|T\|$). Assume P_i is the orthogonal projection of $\mathbf{H}$ onto $\mathbf{M}_i$, so that $S_{ij} = P_i S|_{\mathbf{M}_j}$, and define $V \colon \mathbf{H} \to \mathbf{H} \otimes \mathbf{l}^2$ by $Vx = \sum P_i x \otimes e_i$. We assert that the matrix of $V^* \Psi(S) V$ with respect to $\mathfrak{F}$ is $\operatorname{Schur}_T(S)$. Suppose $m \in \mathbf{M}_j$;

$$(P_i V^* \Psi(S) V)(m) = P_i V^*(S \otimes T)(m \otimes e_j) = P_i V^*(Sm \otimes Te_j) = (P_i \langle Te_j, e_i\rangle S)(m).$$

The last equality follows from the observation that for any $x \in \mathbf{H}$,

$$\begin{aligned}\langle P_i V^*(Sm \otimes Te_j), x\rangle &= \langle Sm \otimes Te_j, VP_i x\rangle \\ &= \langle Sm, P_i x\rangle \langle Te_j, e_i\rangle = \langle P_i \langle Te_j, e_i\rangle Sm, x\rangle.\end{aligned}$$

Thus

$$\|\,\mathrm{Schur}_T\,\|_{\mathrm{cb}} = \|V^*\Psi V\|_{\mathrm{cb}} \le \|\Psi\|_{\mathrm{cb}} \le \|T\|. \quad \square$$

Proof of Theorem 3.3. Assume the hypothesis, so $\bigoplus \mathbf{N}_i \in \operatorname{Lat}\mathfrak{A}$ and the restriction of $\mathfrak{A}$ to $\bigoplus \mathbf{N}_i$ is upper triangular. The matrix of T relative to this decomposition is diagonal, so Φ_{T_0} has a particularly simple form;

$$\Phi_{T_0}(A) = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ 0 & 0 & 2^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} A_{00} & A_{01} & A_{02} & \cdots \\ 0 & A_{11} & A_{12} & \cdots \\ 0 & 0 & A_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & (\frac{1}{2})^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

When the matrices are multiplied out, we are left with

$$\Phi_{T_0}(A) = \begin{pmatrix} A_{00} & \frac{1}{2}A_{01} & \frac{1}{2^2}A_{02} & \frac{1}{2^3}A_{03} & \cdots \\ 0 & A_{11} & \frac{1}{2}A_{12} & \frac{1}{2^2}A_{13} & \cdots \\ 0 & 0 & A_{22} & \frac{1}{2}A_{23} & \cdots \\ 0 & 0 & 0 & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which is a Schur product of A (relative to $\bigoplus \mathbf{N}_i$) with the adjoint of an analytic Toeplitz operator. It follows from Lemma 3.4 that Φ_{T_0} is completely bounded. $\square$

Corollary 3.5. *Assume* $\mathfrak{A}$ *is a norm closed algebra such that* $(\mathfrak{A}\cap\mathfrak{A}^*)''$ *contains a* MASA $\mathfrak{M}$ *and* $\mathfrak{M}\mathfrak{A}\mathfrak{M} \subset \mathfrak{A}$. *Then* $\operatorname{Lat}_{1/2}\mathfrak{A} = \operatorname{Lat}_{\mathrm{cb}}\mathfrak{A}$.

Proof. By Davidson's generalization of the work of Foiaş and Ong (see [3], [6], [12], and [13]), all invariant ranges of $\mathfrak{A}$ are obtained as

$$\mathfrak{R} = \left\{ \sum_{i=0}^{\infty} x_i \middle| x_i \in \mathbf{N}_i,\ \sum_{i=0}^{\infty} (2^i\|x_j\|)^2 < \infty \right\}$$

where $\mathbf{N}_i = \mathbf{M}_i \ominus \mathbf{M}_{i-1}$ $(\mathbf{M}_{-1} = (0);\ i = 0, 1, 2, \ldots)$, and $\{\mathbf{M}_i\}$ is an increasing sequence in $\operatorname{Lat}\mathfrak{A}$. By Theorem 3.3 these are all in $\operatorname{Lat}_{\mathrm{cb}}\mathfrak{A}$. $\square$

4. A REPRESENTATION OF $\operatorname{Lat}_{\mathrm{cb}}\mathfrak{A}$

If $\mathfrak{A}$ is either a C^*-algebra or the algebra of analytic Toeplitz operators, then it is possible to find a Hilbert space $\mathbf{K}$ and a complete contraction $\Phi\colon \mathfrak{A} \to B(\mathbf{K})$ such that

$$\operatorname{Lat}_{\mathrm{cb}}\mathfrak{A} = \{T\mathbf{K} \mid T \in B(\mathbf{K},\mathbf{H})\,;\, AT = T\Phi(A) \text{ for all } A \in \mathfrak{A}\}.$$

When $\mathfrak{A}$ is a C^*-algebra, then $\mathbf{K} = \mathbf{H}$ and Φ is the identity map. When $\mathfrak{A}$ is the algebra of analytic Toeplitz operators, then $\mathbf{K} = \mathbf{H}^{(\infty)}$ and Φ is the infinite inflation map. If $\mathfrak{A}$ is an arbitrary unital norm closed subalgebra of $B(\mathbf{H})$, $\mathbf{K}$ is a Hilbert space, and $\Phi\colon \mathfrak{A} \to B(\mathbf{K})$ is a complete contraction, then define

$$\mathfrak{L}_\Phi \equiv \{T\mathbf{K} \mid T \in B(\mathbf{K},\mathbf{H}),\, AT = T\Phi(A) \text{ for all } A \in \mathfrak{A}\}.$$

It follows immediately that $\mathfrak{L}_\Phi \subset \operatorname{Lat}_{\mathrm{cb}}\mathfrak{A}$. We now prove that Φ may always be chosen so that equality holds.

Lemma 4.1. *Assume $\mathfrak{A}$ is a norm closed unital subalgebra of $B(\mathbf{H})$, $T \neq 0$, and $T\mathbf{H} \in \operatorname{Lat_{cb}} \mathfrak{A}$. Then there exists a Hilbert space $\mathbf{K}$ and $S \in B(\mathbf{K}, \mathbf{H})$ such that $\operatorname{ran} S = \operatorname{ran} T$ and $\|\Phi_S\|_{\mathrm{cb}} = 1$.*

Proof. Since $\Phi_{T_0}: \mathfrak{A} \to B((\ker T)^{\perp})$ is a completely bounded unital homomorphism, Theorem 8.1 of [16] gives us an invertible operator $R \in B((\ker T)^{\perp})$ such that the map $\pi: \mathfrak{A} \to B((\ker T)^{\perp})$ defined by $\pi(A) = R^{-1}\Phi_{T_0}(A)R$ is a complete contraction (see also [15]). Letting $S = T_0 R$ and $\mathbf{K} = (\ker T)^{\perp}$, we have

$$\pi(A) = R^{-1}\Phi_{T_0}(A)R = R^{-1}T_0^{-1}AT_0R = (T_0R)^{-1}A(T_0R) = \Phi_S(A).$$

It follows that $\|\Phi_S\|_{\mathrm{cb}} \leq 1$, and since Φ_S is unital, $\|\Phi_S\|_{\mathrm{cb}} = 1$. □

Theorem 4.2. *Assume $\mathfrak{A}$ is a norm closed unital subalgebra of $B(\mathbf{H})$. Then there exists a Hilbert space $\mathbf{K}$ and a complete contraction $\Phi: \mathfrak{A} \to B(\mathbf{K})$ such that $\operatorname{Lat_{cb}} \mathfrak{A} = \mathfrak{L}_{\Phi}$.*

Proof. For each $\mathfrak{R} \in \operatorname{Lat_{cb}} \mathfrak{A}$, $\mathfrak{R} \neq (0)$, choose a Hilbert space $\mathbf{H}_{\mathfrak{R}}$ and $S_{\mathfrak{R}} \in B(\mathbf{H}_{\mathfrak{R}}, \mathbf{H})$ so that $\operatorname{ran} S_{\mathfrak{R}} = \mathfrak{R}$ and $\|\Phi_{S_{\mathfrak{R}}}\|_{\mathrm{cb}} = 1$ (as in Lemma 4.1); henceforth we write $\Phi_{\mathfrak{R}}$ in place of $\Phi_{S_{\mathfrak{R}}}$. Let $\Omega = \operatorname{Lat_{cb}} \mathfrak{A} \backslash \{(0)\}$ and define

$$\Phi \equiv \bigoplus_{\mathbf{R} \in \Omega} \Phi_{\mathfrak{R}} \quad \text{and} \quad \mathbf{K} \equiv \bigoplus_{\mathfrak{R} \in \Omega} \mathbf{H}_{\mathfrak{R}}.$$

It follows from a "canonical shuffle" (as discussed in [16]) that Φ is completely bounded and $\|\Phi\|_{\mathrm{cb}} = 1$.

Suppose $\mathfrak{R}_0 \in \operatorname{Lat_{cb}} \mathfrak{A}$; it is clear that $(0) \in \mathfrak{L}_{\Phi}$, so assume $\mathfrak{R}_0 \neq (0)$. Define $T: \mathbf{K} \to \mathbf{H}$ by $T(\bigoplus x_{\mathfrak{R}}) = S_{\mathfrak{R}_0}(x_{\mathfrak{R}_0})$; T is clearly bounded and if $A \in \mathfrak{A}$,

$$AT(\bigoplus x_{\mathfrak{R}}) = AS_{\mathfrak{R}_0}(x_{\mathfrak{R}_0}) = S_{\mathfrak{R}_0}\Phi_{\mathfrak{R}_0}(A)(x_{\mathfrak{R}_0}) = T\Phi(A)(\bigoplus x_{\mathfrak{R}}).$$

Thus $T\mathbf{K} \in \mathfrak{L}_{\Phi}$, and clearly $T\mathbf{K} = \mathfrak{R}_0$. □

The following corollary appears in [11] when $\mathfrak{A}$ is the algebra of analytic Toeplitz operators and Φ is the infinite inflation map. The proof is obtained by combining Theorems 2.2 and 4.2.

Corollary 4.3. *Assume $\mathfrak{A}$ is a norm closed unital subalgebra of $B(\mathbf{H})$ and $\Phi: \mathfrak{A} \to B(\mathbf{K})$ is a complete contraction such that $\mathfrak{L}_{\Phi} = \operatorname{Lat_{cb}} \mathfrak{A}$. Then*

$$\operatorname{Lat}_{1/2} \mathfrak{A} \otimes \mathfrak{K} = \{\mathfrak{R} \otimes 1 | \mathfrak{R} \in \mathfrak{L}_{\Phi}\}.$$

5. Questions

The correct generalization of Dixmier's question to arbitrary unital norm closed operator algebras is now obvious: does $\operatorname{Lat}_{1/2} \mathfrak{A} = \operatorname{Lat_{cb}} \mathfrak{A}$? It seems most likely that examples where equality does not hold will be discovered.

In view of Theorem 4.2, one might ask for a simple complete contraction with which to write $\mathfrak{L}_{\Phi} = \operatorname{Lat_{cb}} \mathfrak{A}$, as we are able to do when $\mathfrak{A}$ is a C^*-algebra or the algebra of analytic Toeplitz operators. For example, if $\mathfrak{A}$ is the adjoint of the algebra of analytic Toeplitz operators, can a simple Φ be found such that $\mathfrak{L}_\Phi = \mathrm{Lat}_{\mathrm{cb}}\,\mathfrak{A}$? The infinite inflation clearly does not work in this situation. The ranges of Hankel operators are invariant under this algebra, and the transpose map intertwines the algebra with the Hankel operators ($AH = HA^T$ for all $A \in \mathfrak{A}$ and for all Hankel operators). One can prove that all of the closed invariant subspaces of $\mathfrak{A}$ are obtained as ranges of Hankel operators. Since the restriction of the transpose map to this algebra is completely isometric, the ranges of Hankel operators are all in $\mathrm{Lat}_{\mathrm{cb}}\mathfrak{A}$. Is $\mathrm{Lat}_{\mathrm{cb}}\mathfrak{A}$ the lattice spanned by the ranges of Hankel operators together with the ranges in L_∞?

In [9] and [11], examples are given of algebras $\mathfrak{A}$ such that $\mathrm{Lat}_{\mathrm{cb}}\,\mathfrak{A}$ consists of all operator ranges that may be written as $P\mathbf{M}$, where $P \in B(\mathbf{H}^{(\infty)}, \mathbf{H})$ is the partial isometry defined by $P(\bigoplus x_i) = x_0$, and $\mathbf{M} \in \mathrm{Lat}\,\mathfrak{A}^{(\infty)}$. Is there a more general principle at work here?

References

1. E. Azoff, *Invariant linear manifolds and the selfadjointness of operator algebras*, Amer. J. Math. **99** (1977), 121–138.
2. A. Connes, *Classification of injective factors*, Ann. of Math. **104** (1976), 73–115.
3. K. R. Davidson, *Invariant operator ranges for reflexive algebras*, J. Operator Theory **7** (1982), 101–108.
4. J. Dixmier, *Étude sur les variétés et les opérateurs de Julia*, Bull. Soc. Math. France **77** (1949), 11–101.
5. P. A. Fillmore and J. P. Williams, *On operator ranges*, Adv. in Math. **7** (1971), 254–281.
6. C. Foiaş, *Invariant para-closed subspaces*, Indiana Univ. Math. J. **20** (1972), 897–900.
7. U. Haagerup, *Solution of the similarity problem for cyclic representations of C^*-algebras*, Ann. of Math. **118** (1983), 215–240.
8. G. W. Mackey, *On the domains of closed linear transformations in Hilbert space*, Bull. Amer. Math. Soc. **52** (1946), 1009.
9. B. Mathes, *On invariant operator ranges of abelian strictly cyclic algebras*, J. Operator Theory, (to appear).
10. E Nordgren, M. Radjabalipour, H. Radjavi, and P. Rosenthal, *On invariant operator ranges*, Trans. Amer. Math. Soc. **251** (1979), 389–398.
11. E. Nordgren and C. Roy, *On invariant ranges of the analytic Toeplitz operators*, Indiana Univ. Math. J. (to appear).
12. S.-C. Ong, *Operator algebras and invariant operator ranges*, Ph. D. dissertation, Dalhousie University, Halifax, Nova Scotia, 1979.
13. ——, *Invariant operator ranges of nest algebras*, J. Operator Theory **3** (1980), 195–201.
14. V. I. Paulsen, *Completely bounded maps on C^*-algebras and invariant operator ranges*, Proc. Amer. Math. Soc. **86** (1982), 91–96.
15. ——, *Every completely polynomially bounded operator is similar to a contraction*, J. Funct. Anal. **55** (1984), 1–17.

16. ——, *Completely bounded maps and dilations*, Pitman Research Notes in Math. Series, no. 146, Wiley, New York, 1986.

17. C. Roy, *Operator ranges of shifts and C^*-algebras*, Ph. D. dissertation, University of New Hampshire, Durham, N. H., 1986.

18. D. Voiculescu, *Sur les sous-espaces parafermés invariants d'une algèbre de von Neumann*, Bull. Sci. Math. (2) **96** (1972), 161–168.

DEPARTMENT OF MATHEMATICS, DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, CANADA B3H 3J5