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COMPLEMENTED COPIES OF c_0 IN VECTOR-VALUED HARDY SPACES

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ABSTRACT. Let X be a complex Banach space containing a copy of c_0, let $\mathbb{T}$ be the unit circle and let D be the open unit disk in the complex plane. Then $H^p(\mathbb{T}, X)$ contains a complemented copy of c_0 for $1 \leq p < \infty$. The corresponding result for $H^p(D, X)$ fails for $1 \leq p \leq \infty$.

1. INTRODUCTION

If X is a Banach space which contains a copy of c_0 then $L^p([0,1], X)$ contains a complemented copy of c_0 for $1 \leq p < \infty$ [5]. In this note we consider the corresponding problem for vector-valued Hardy spaces. However, there are two natural Hardy spaces to consider, $H^p(\mathbb{T}, X)$ and $H^p(D, X)$. We will show that $H^p(\mathbb{T}, X)$ contains a complemented copy of c_0 whenever $1 \leq p < \infty$ and X is a complex Banach space containing a copy of c_0. The proof will allow us to extend the result to a slightly larger class of spaces. We will also show that the spaces $H^p(D, \ell_\infty)$ do not contain complemented copies of c_0 for $1 \leq p \leq \infty$.

2. PRELIMINARIES AND RESULTS

Throughout this note $\mathbb{T}$ will denote the unit circle, $\frac{d\theta}{2\pi}$ will denote normalized Lebesgue measure on $\mathbb{T}$, and D will be the open unit disk in the complex plane.

Let X be a complex Banach space and let $1 \leq p \leq \infty$. The space $H^p(D, X)$ is the collection of all X-valued analytic functions on D with $\|f\|_p < \infty$ where

$$\|f\|_p = \sup_{0 \leq r < 1} \left\{ \int_0^{2\pi} \|f(re^{i\theta})\|^p \frac{d\theta}{2\pi} \right\}^{1/p}$$

for $1 \leq p < \infty$, and

$$\|f\|_\infty = \sup_{z \in D} \|f(z)\|.$$

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If $f\colon \mathbb{T} \to X$ is a vector-valued function then its Fourier coefficients are

$$\hat{f}(n) = \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}, \quad \text{for each } n \in \mathbb{Z}.$$

For $1 \le p \le \infty$, we define

$$H^p(\mathbb{T}, X) = \{f \in L^p(\mathbb{T}, X) \colon \hat{f}(n) = 0 \quad \text{for all } n < 0\}.$$

Before we get to the main result we need a lemma which appears implicitly in [2] and [7].

Lemma. *If a Banach space X contains a sequence $(x_n)_{n=1}^{\infty}$ which is equivalent to the unit vector basis of c_0 and if $(x_n^*)_{n=1}^{\infty}$ is a weak* null sequence in X^* such that $\inf_n |x_n^*(x_n)| > 0$, then X contains a complemented copy of c_0.*

Proof. Define an operator $S\colon X \to c_0$ by $S(x) = (x_n^*(x))_{n=1}^{\infty}$. Clearly, S is well defined, since $(x_n^*)_{n=1}^{\infty}$ is weak* null, and also bounded and linear. The series $\sum_{n=1}^{\infty} x_n$ is weakly unconditionally Cauchy but $\sum_{n=1}^{\infty} S(x_n)$ is not unconditionally convergent in c_0 because $\inf_n |x_n^*(x_n)| > 0$. By [1] there is a subsequence $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $(y_n)_{n=1}^{\infty}$ is equivalent to the unit vector basis of c_0 and $S|_{[y_n]_{n=1}^{\infty}}$ is an isomorphism of $[y_n]_{n=1}^{\infty}$ onto $Y = [S(y_n)]_{n=1}^{\infty}$. Y is a subspace of c_0 which is isomorphic to c_0 and so is complemented in c_0 by a bounded linear projection Q (see [8]). Consider the operator $P\colon X \to X$ defined by $P(x) = (S|_{[y_n]_{n=1}^{\infty}})^{-1} QS(x)$ for $x \in X$. P is a bounded linear projection of X onto $[y_n]_{n=1}^{\infty}$, and since $[y_n]_{n=1}^{\infty}$ is isomorphic to c_0, the proof is complete.

Theorem. *Let X be a complex Banach space and $1 \le p < \infty$. If X contains a copy of c_0, then $H^p(\mathbb{T}, X)$ contains a complemented copy of c_0.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X equivalent to the unit vector basis of c_0. Then there are constants K_1, $K_2 > 0$ so that for any choice of scalars $a_1, a_2, \dots, a_n$,

$$K_1 \max_{1 \le j \le n} |a_j| \le \|\sum_{j=1}^{n} a_j x_j\| \le K_2 \max_{1 \le j \le n} |a_j|.$$

For each $n \in \mathbb{N}$ define $y_n \in H^p(\mathbb{T})$ by $y_n(e^{i\theta}) = e^{in\theta}$. Then $x_n \otimes y_n \in H^p(\mathbb{T}, X)$, where $(x_n \otimes y_n)(e^{i\theta}) = x_n e^{in\theta}$ and

$$K_1 \max_{1 \le j \le n} |a_j| \le \|\sum_{j=1}^{n} a_j (x_j \otimes y_j)(e^{i\theta})\| \le K_2 \max_{1 \le j \le n} |a_j|.$$

Therefore

$$K_1 \max_{1 \le j \le n} |a_j| \le \|\sum_{j=1}^{n} a_j (x_j \otimes y_j)\|_p \le K_2 \max_{1 \le j \le n} |a_j|.$$

That is, $(x_n \otimes y_n)_{n=1}^{\infty}$ is equivalent to the unit vector basis of c_0 in $H^p(\mathbb{T}, X)$. Now let $(x_n^*)_{n=1}^{\infty}$ be a bounded sequence in X^* which is biorthogonal to $(x_n)_{n=1}^{\infty}$

and let $(y_n^*)_{n=1}^{\infty}$ be a sequence in $L^{\infty}(\mathbb{T})$ defined by $y_n^*(e^{i\theta}) = e^{-in\theta}$. Clearly, $(x_n^* \otimes y_n^*)_{n=1}^{\infty}$ is a sequence in $(H^p(\mathbb{T}, X))^*$, and for each $n \in \mathbb{N}$, $(x_n^* \otimes y_n^*)(x_n \otimes y_n) = 1$. Also, if $f \in H^p(\mathbb{T}, X)$, then $(x_n^* \otimes y_n^*)(f) = x_n^*(\hat{f}(n))$ and $x_n^*(\hat{f}(n)) \to 0$ as $n \to \infty$, since $\|\hat{f}(n)\| \to 0$ as $n \to \infty$. To see this, define $S_f \colon L^{\infty}(\mathbb{T}) \to X$ by

$$S_f(g) = \int_0^{2\pi} g(e^{i\theta}) f(e^{i\theta}) \frac{d\theta}{2\pi} \quad \text{for } g \in L^{\infty}(\mathbb{T}).$$

S_f is a compact linear operator [3], so $\{\hat{f}(n)\}_{n=1}^{\infty} = \{S_f(e^{-in\theta})\}_{n=1}^{\infty}$ is a relatively compact subset of X. If $x^* \in X^*$, then

$$x^*(\hat{f}(n)) = x^* S_f(e^{-in\theta}) = \int_0^{2\pi} x^* f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi} \to 0$$

as $n \to \infty$ since $x^* f \in L^p(\mathbb{T})$ and the Riemann-Lebesgue lemma. Therefore $(\hat{f}(n))_{n=1}^{\infty}$ converges weakly to 0 and hence converges to 0 in norm.

Thus $(x_n^* \otimes y_n^*)_{n=1}^{\infty}$ is weak* null so $(x_n \otimes y_n)_{n=1}^{\infty}$ and $(x_n^* \otimes y_n^*)_{n=1}^{\infty}$ satisfy the conditions of the lemma, which completes the proof.

Remark 1. It is clear that this proof can be used in the following setting: Let G be a compact abelian group with normalized Haar measure on G. Let $\hat{G}$ be the dual group of G, and let Λ be a subset of $\hat{G}$. For $1 \le p \le \infty$ and a complex Banach space X, we define

$$L^p_{\Lambda}(G, X) = \{f \in L^p(G, X) \colon \hat{f}(\gamma) = 0 \quad \text{for all } \gamma \notin \Lambda\}.$$

If X contains a copy of c_0, if $1 \le p < \infty$, and if Λ is infinite, then $L^p_{\Lambda}(G, X)$ contains a complemented copy of c_0. Note that if $f \in L^1(G)$, then the net $(\hat{f}(\gamma))_{\gamma \in \Lambda}$ is an element of $c_0(\Lambda)$ (see [6]).

Remark 2. The conclusion of the theorem does not hold true if $H^p(\mathbb{T}, X)$ is replaced by $H^p(D, X)$. For example, consider $H^p(D, \ell_{\infty})$ for $1 \le p \le \infty$. By a result of Dowling [4], $H^p(D, \ell_{\infty})$ is a dual Banach space for $1 \le p \le \infty$. However, Bessaga and Pelczynski [1] have proved that c_0 is never complemented in the dual of a Banach space. Therefore, $H^p(D, \ell_{\infty})$ does not contain complemented copies of c_0. We know that $H^p(\mathbb{T}, \ell_{\infty})$ is isomorphic to a subspace of $H^p(D, \ell_{\infty})$, so the results of this note show that $H^p(\mathbb{T}, \ell_{\infty})$ is not isomorphic to a complemented subspace of $H^p(D, \ell_{\infty})$ when $1 \le p < \infty$.

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