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IRREDUCIBLE REPRESENTATIONS OF NORMAL SPACES

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ABSTRACT. We define the notion of irreducible polyhedral representation of a normal space making use of approximate inverse systems. This generalizes the concept of irreducible polyhedral expansions introduced in 1937 by Freudenthal for metric compacta and generalized to uniform spaces by Isbell in 1961. We show that every normal space X has an irreducible polyhedral representation whose dimension is dim X and whose weight is weight (X). Approximate inverse systems were first introduced by S. Mardešić and this author. The concept generalizes that of inverse system and was essentially used in proving that each Hausdorff compactum of integral cohomological dimension $\leq n$ is the cell-like image of a Hausdorff compactum of covering dimension $\leq n$.

1. INTRODUCTION

In [M-R 1] we defined a new concept called approximate (inverse) system, which generalizes that of the classical (commutative) inverse system. We used the new notion to characterize covering dimension (dim) for compact Hausdorff spaces. Then in [M-R 2] we proved, relying heavily on the machinery of approximate systems, that if X is a compact Hausdorff space whose cohomological dimension $\dim_{\mathbb{Z}} X \leq n$, then there is a compact Hausdorff space Y, $\dim Y \leq n$, and a cell-like map of Y onto X.

Now we are going to explore the use of approximate systems in developing irreducible polyhedral representations of normal spaces in accordance with their dimension. This notion, to be defined below, is a type of generalization of irreducible polyhedral expansion first introduced for compact metric spaces in [Fr] and which was extended to the case of uniform spaces by J.R. Isbell [Is] and studied for compact Hausdorff spaces in [M1]. Since it is not generally possible to have a polyhedral expansion (in terms of finite polyhedra) for normal spaces, an entirely new concept is needed. This is why we shall speak of "representation" of a normal space, and as will be demonstrated, approximate systems are the perfect vehicles for such representations.

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This work uses techniques developed in [M-R 1], [M-R 2]. In particular our Lemma 1 (§3) is much like a lemma in [M-R 2] and our proof of the Theorem in this paper is developed along the lines of a proof in [M-R 1]. In fact, this research is an outgrowth of that done in [M-R 1], [M-R 2].

2. Approximate systems and representations

In this paper assume all spaces are Hausdorff and all polyhedra are compact, i.e., have finite triangulations. Dimension, dim, is defined in terms of finite open covers and finite open refinements ([En] or [Na]). By map we mean continuous function.

Definition 1. An *approximate* (*inverse*) *system* of metric compacta $\mathbf{X} = (X_a, \varepsilon_a, p_{aa'}, A)$ consists of the following: A directed ordered set $(A, \leq)$ with no maximal element; for each $a \in A$, a compact metric space X_a with metric $d = d_a$ and a real number $\varepsilon_a > 0$; for each pair $a \leq a'$ from A, a map $p_{aa'}: X_{a'} \to X_a$, satisfying the following conditions:

(A1) $d(p_{a_1a_2}p_{a_2a_3}, p_{a_1a_3}) \leq \varepsilon_{a_1}$, $a_1 \leq a_2 \leq a_3$; $p_{aa} = id$.

(A2) $(\forall a \in A)(\forall \eta > 0)(\exists a' \geq a)(\forall a_2 \geq a_1 \geq a')$
$d(p_{aa_1}p_{a_1a_2}, p_{aa_2}) \leq \eta$.

(A3) $(\forall a \in A)(\forall \eta > 0)(\exists a' \geq a)(\forall a'' \geq a')(\forall x, x' \in X_{a''})$
$d(x, x') \leq \varepsilon_{a''} \Rightarrow d(p_{aa''}(x), p_{aa''}(x')) \leq \eta$.

We refer to the numbers ε_a as the *meshes* of $\mathbf{X}$.

If $\pi_a: \prod_{a\in A} X_a \to X_a$, $a \in A$, denote projections, we define the limit space $X = \lim \mathbf{X}$ and the natural projections $p_a: X \to X_a$ as follows.

Definition 2. A point $x = (x_a) \in \Pi X_a$ belongs to $X = \lim \mathbf{X}$ provided for every $a \in A$,

$$x_a = \lim_{a_1} p_{aa_1}(x_{a_1}).$$

The natural projection $p_a = \pi_a|X: X \to X_a$.

Definition 3. We say $\{X_a, p_{aa'}, A\}$ is an *almost commutative system* [M-S] if there exist numbers $\varepsilon_a > 0$, $a \in A$, so that $\{X_a, \varepsilon_a, p_{aa'}, A\}$ is an approximate system.

Definition 4. Let K be a complex and let $f, g: X \to |K|$ be maps. We say that g is a *K-modification* of f if for every $x \in X$ and $\sigma \in K$, $f(x) \in \sigma$ implies $g(x) \in \sigma$. Note that a simplicial approximation $\varphi: K_1 \to K_2$ of a map $\pi: |K_1| \to |K_2|$ is a K_2-modification of π. Moreover, if K' is a subdivision of K, and $g: X \to |K'|$ is a K'-modification of $f: X \to |K'| = |K|$, then g is also a K-modification of f.

Definition 5. A map $f: X \to |K|$ is *K-irreducible* if for every K-modification g of f, one has $g(X) = |K|$. Since f is its own K-modification, a K-irreducible map f is onto. A map $f: X \to P$ where P is a polyhedron is called *irreducible* if it is K-irreducible for some triangulation K of P. Note that every irreducible map $f: X \to P$ is onto.

We now come to our definition of irreducible polyhedral representation.

Definition 6. Let X be a normal space and $\mathbf{P} = \{P_a, p_{aa'}, A\}$ be an almost commutative system of compacta P_a. A set of maps $f_a: X \to P_a$, $a \in A$, is said to be a *representation* of X in $\mathbf{P}$ if the map $f: X \to \prod_{a \in A} P_a$ given by $f(x) = (f_a(x))$ embeds X onto a dense subspace of $\lim \mathbf{P}$. We say the representation is of *dimension* $\leq n$ if $\dim P_a \leq n$ for all a, and we define its *weight* to be card(A). We call it *cofinite* if the indexing set A is cofinite, i.e., for each $a \in A$ there are only finitely many $a' \in A$ with $a' \leq a$.

We call the representation *polyhedral* if each P_a is a polyhedron and *irreducible* if in addition each $f_a: X \to P_a$ is irreducible and for $a \leq a'$, $p_{aa'}$ is irreducible. A polyhedral representation will be called *simplicially irreducible* if also for each a there is a fixed triangulation K_a of P_a so that f_a is K_a-irreducible and whenever $a \leq a'$, then $p_{aa'}$ carries $K_{a'}$ simplicially to a subdivision of K_a and $p_{aa'} \circ f_{a'}$ is a K_a-modification of $f_a: X \to P_a$.
Note. If f_a is K_a-irreducible, then the latter condition is readily seen to imply that $p_{aa'}$ is K_a-irreducible also. Hence simplicially irreducible implies irreducible.

Our main theorem is the following.

Theorem. *Every normal space X has a cofinite simplicially irreducible representation of dimension $\leq \dim X$, and of weight $\leq$ weight(X).*

We delay proof until §4. The following Corollary appears as Corollary 3 in [M1] and elsewhere.

Corollary. *If X is a normal space and* $\dim X \leq n$, *then X has a Hausdorff compactification P such that* $\dim P \leq n$.
Proof. Let $P = \lim \mathbf{P}$ and apply Theorems 2 and 4 of [M-R 1].

3. Basic lemma

The next lemma is basic to the later construction. It is similar to Lemma 2 of [M1] and Lemma 1 of [M-R 2].

Lemma 1. *Let X be a normal space, let $f_i: X \to P_i = |K_i|$ be maps to triangulated polyhedra, and let $\varepsilon_i > 0$, $i = 1, \cdots, k$. Then there exist a polyhedron $Q = |L|$, $\dim Q \leq \dim X$, an L-irreducible map $g: X \to Q$, and maps $p_i: Q \to P_i$ which are simplicial from L to some subdivision L_i of K_i such that $d(f_i, p_i g) \leq \varepsilon_i$ and $p_i g$ is an L_i-modification of f_i, $i = 1, \cdots, k$. The subdivision L_i may be given arbitrarily fine mesh. Moreover, if for a given index i the map f_i is K_i-irreducible, then the corresponding map p_i is L_i-irreducible.*
Proof. Let L_i be a subdivision of K_i with

(1) $$\text{mesh } L_i \leq \varepsilon_i/2.$$

Note that f_i is L_i-irreducible if it is K_i-irreducible. Let $P = P_1 \times \cdots \times P_k$, let $f = f_1 \times \cdots \times f_k \colon X \to P$, and let $\pi_i \colon P \to P_i$, $i = 1, \cdots, k$, be the projections. Choose $\delta > 0$ so small that

(2) $$d(x, x') \leq \delta \Rightarrow d(\pi_i(x), \pi_i(x')) \leq \varepsilon_i/2, \; i = 1, \cdots, k.$$

Let K be a triangulation of P so fine that

(3) $$\text{mesh } K \leq \delta,$$

and the projections $\pi_i \colon |K| \to |L_i|$ admit simplicial approximations $\overline{p}_i \colon K \to L_i$, $i = 1, \cdots, k$. Since $\overline{p}_i$ is an L_i-modification of π_i, we have

(4) $$d(\overline{p}_i, \pi_i) \leq \text{ mesh } L_i \leq \varepsilon_i/2.$$

Let $\mathscr{U}$ be the (finite) open cover of X consisting of the sets $f^{-1}(st(v))$, v a vertex of K. Choose a finite open cover $\mathscr{V}$ of X so that $\mathscr{V}$ refines $\mathscr{U}$ and so that $\dim(\mathscr{N}) \leq \dim X$, where $\mathscr{N}$ is the nerve of $\mathscr{V}$. Let $g_0 \colon X \to |\mathscr{N}|$ be a canonical map.

There is a simplicial map $h_0 \colon \mathscr{N} \to K$ defined on vertices of $\mathscr{N}$ by sending $V \in \mathscr{V}$ to a vertex v of K such that $V \subset f^{-1}(st(v))$. There is a subcomplex L of $\mathscr{N}$ and an $\mathscr{N}$-modification $g \colon X \to |L|$ of g_0 such that g is L-irreducible. This follows by choosing a minimal subcomplex L of $\mathscr{N}$ for which there is an $\mathscr{N}$-modification g into $|L|$ and then applying Lemma 2 below.

Put $Q = |L|$ and note that $\dim Q \leq \dim |\mathscr{N}| \leq \dim X$. Let $h = h_0|Q \colon Q \to |K|$; we see that $h \colon L \to K$ is simplicial. Choose p_i to be $\overline{p}_i h \colon Q \to |L_i| = P_i$. Then $p_i \colon L \to L_i$ is simplicial.

Note that $d(f, hg) \leq$ mesh $K \leq \delta$, and therefore,

(5) $$d(\pi_i f, \pi_i hg) = d(f_i, \pi_i hg) \leq \varepsilon_i/2, \; i = 1, \cdots, k.$$

From (4) we get $d(\overline{p}_i h, \pi_i h) \leq \varepsilon_i/2$. But $\overline{p}_i h = p_i$, so we have,

(6) $$d(p_i, \pi_i h) \leq \varepsilon_i/2.$$

Hence,

(7) $$d(p_i g, \pi_i hg) \leq \varepsilon_i/2.$$

Now (5) and (7) yield

(8) $$d(f_i, p_i g) \leq \varepsilon_i.$$

Next we will show that if f_i is K_i-irreducible, hence L_i-irreducible, then p_i is L_i-irreducible. If we can show that $p_i g$ is L_i-irreducible, then certainly p_i will be L_i-irreducible. It is sufficient to show that $p_i g$ is an L_i-modification of f_i.

Choose $x \in X$ and suppose $f_i(x) = \pi_i f(x)$ lies in the interior of the simplex σ of L_i. The choices of g and h are such that if for a simplex τ of K, $f(x)$ lies in τ, then $hg(x) \in \tau$. Since $\overline{p}_i$ is a simplicial approximation of π_i, then $\overline{p}_i(f(x)) \in \sigma$ and hence $\overline{p}_i(\tau) \subset \sigma$. Thus $\overline{p}_i f(x)$ and $\overline{p}_i hg(x) = p_i g(x)$ lie in σ. This shows that both $f_i(x)$ and $p_i g(x)$ lie in σ, so $p_i g$ is an L_i-modification of f_i and the proof is complete.

A map $g: X \to |L|$ is said to be *essential* on a simplex σ of L if there is no map $f: g^{-1}(\sigma) \to \partial\sigma$ such that f agrees with g on $g^{-1}(\partial\sigma)$.

Lemma 2. *Let X be a normal space, L a finite complex, and $g: X \to |L|$ be a map such that no L-modification of g carries X into a proper subcomplex of L. Then g is L-irreducible.*

Proof. It is sufficient to show that g is surjective; for every L-modification of g satisfies the hypotheses of this lemma.

Suppose λ is a principal simplex of L. Then g must be essential on λ or else there would be an L-modification of g whose image contained no points of the interior of λ. But as argued in the proof of 3.6 of [R-S], this implies that g is essential on every simplex of L and hence g maps onto the interior of every simplex of L. This proves that g is surjective.

Note. Lemma 2 is still true for infinite complexes for which each simplex lies in a principal simplex, e.g., locally finite dimensional complexes.

4. Proof of theorem

The reader will notice many similarities between the forthcoming proof and the proof of Theorem 5 of [M-R 1]. We shall avoid repetition where possible, but some redundancy is necessary to maintain the integrity of the current proof.

As in [M-R 1], choose an embedding e of X into a Tihonov cube $Y = I^\tau$ where $\tau = \text{weight}(X)$ and take Y to be the inverse limit of an (commutative) inverse system $(Y_a, q_{aa'}, A)$ of finite dimensional cubes where A is cofinite with no maximal element and $\text{card}(A) \leq \text{weight}(X)$. For simplicity, assume $X \subset Y$, i.e., that e is the inclusion map. Let $q_a: Y \to Y_a$ be the natural projections and let $|a| \geq 0$ denote the number of predecessors of $a \in A$. Using Lemma 1 of the current paper instead of Lemma 5 of [M-R 1], obtain the following data: for each $a \in A$, a compact triangulated polyhedron $P_a = |K_a|$, $\dim P_a \leq \dim X$, maps $f_a: X \to P_a$, $h_a: P_a \to Y_a$ and numbers $\varepsilon_a > 0$, $\delta_a > 0$ and for each pair $a \leq a'$ a map $p_{aa'}: P_{a'} \to P_a$. We require that each $f_a: X \to P_a$ be K_a-irreducible and that the following conditions hold (see 5.1 of [M-R 1]).

1. $d(p_{aa'}f_{a'}, f_a) \leq \frac{\varepsilon_a}{3^{|a'|-|a|}}, a < a'$, $p_{aa} = id$.
2. $d(q_a|X, h_a f_a) \leq \delta_a/3$,
3. $x, x' \in P_a, d(x, x') \leq \varepsilon_a \Rightarrow d(h_a(x), h_a(x') \leq \delta_a/3$,
4. $x, x' \in P_{a'}, d(x, x') \leq \varepsilon_{a'} \Rightarrow d(p_{aa'}(x), p_{aa'}(x')) \leq \frac{\varepsilon_a}{3^{|a'|-|a|}}$,
5. $y, y' \in Y_a, d(y, y') \leq \delta_{a'} \Rightarrow d(q_{aa'}(y), q_{aa'}(y')) \leq \frac{\delta_a}{3^{|a'|-|a|}}$,
6. $p_{aa'}$ carries $K_{a'}$ simplicially to a subdivision $L_a^{a'}$ of K_a, $p_{aa'}f_{a'}$ is an $L_a^{a'}$-modification of f_a, and mesh $(L_a^{a'}) \leq \frac{1}{|a'|+1}$, $a \leq a'$.

The verification that $\mathbf{P} = (P_a, \varepsilon_a, p_{aa'}, A)$ is an approximate system is precisely as in 5.2 of [M-R 1]. Consider the map $f: X \to \prod_{a\in A} P_a$ given by

$f(x) = (f_a(x))$. We claim that each $f(x)$ is a thread of $\mathbf{P}$. It is necessary to show that for a given $a \in A$,

7. $f_a(x) = \lim_{a'} p_{aa'} f_{a'}(x)$.

However, it is clear that $\lim_{a' \geq a} |a'| = \infty$. Therefore an application of 6 yields 7.

The injectivity of f goes as in 5.4 of [M-R 1] and that f carries X onto a dense subspace of $\lim \mathbf{P}$ is obtained from the proof of 5.5 therein (note though, that you should set $f_a = g_a$). To show that f is an embedding, we shall show that f is a closed map of X to $f(X)$.

Let B be closed in X and suppose $x \in X \backslash B$. We shall find an $a' \in A$ such that

8. $d(f_{a'}(b), f_{a'}(x)) > \varepsilon_{a'}$ for all $b \in B$.

Hence with U as the $\varepsilon_{a'}$-neighborhood of $f_{a'}(x)$ in $P_{a'}$, we have $U \cap f_{a'}(B) = \phi$. Then by Lemma 3 of [M-R 1], $p_{a'}^{-1}(U)$ is a neighborhood of $f(x)$ in $\lim \mathbf{P}$ which does not intersect $f(B)$.

Note that $x \notin \overline{B}$ (closure in Y). Hence there exists $a \in A$ and $\delta > 0$ such that $d(q_a(\overline{B}), q_a(x)) > \delta$. Thus $d(q_a(B), q_a(x)) > \delta$. Choose k so large that $\frac{\delta_a}{3^{k-|a|}} < \delta$. Select $a' \in A$ so that $|a'| \geq k$ and $a' \geq a$. Then 5 and the fact that $(Y_a, q_{aa'}, A)$ is a commutative system show us that

9. $d(q_{a'}(b), q_{a'}(x)) > \delta_{a'}$ for all $b \in B$.

Suppose it were true that $d(f_{a'}(b), f_{a'}(x)) \leq \varepsilon_{a'}$ for some $b \in B$. This and 3 would imply,

10. $d(h_{a'} f_{a'}(b), h_{a'} f_{a'}(x)) \leq \delta_{a'}/3$.

But 2 yields,

11. $d(q_{a'}(x), h_{a'} f_{a'}(x)) \leq \delta_{a'}/3$ and

12. $d(q_{a'}(b), h_{a'} f_{a'}(b)) \leq \delta_{a'}/3$.

Using 12, 10, 11, we would get, $d(q_{a'}(b), q_{a'}(x)) \leq \delta_{a'}$, which contradicts 9. Hence we conclude,

13. $d(f_{a'}(b), f_{a'}(x)) > \varepsilon_{a'}$ for all $b \in B$.

The proof is now complete.

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