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SIMPLE C^*-ALGEBRAS AND SUBGROUPS OF Q

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ABSTRACT. A special case of a conjecture of R. Douglas is solved by an elementary argument using K_0-theory.

Let Γ be a subgroup of the additive reals $\mathbf{R}$ and let $\Gamma^+ = \{x \in \Gamma: x \geq 0\}$. Douglas [2] defines a *one-parameter semigroup of isometries* to be a homomorphism $x \mapsto V_x$ of Γ^+ into the set of isometries on some Hilbert space H (i.e. $V_{x+y} = V_x V_y$ for $x, y \in \Gamma^+$ and $V_0 = 1$). Denoting by $A_\Gamma(V_x)$ the C^*-algebra generated by all V_x $(x \in \Gamma^+)$ and calling the map $x \mapsto V_x$ *nonunitary* if no V_x is unitary except $V_0 = 1$, he shows that if $x \mapsto V_x$ and $x \mapsto W_x$ are nonunitary one-parameter semigroups of isometries on Γ then the algebras $A_\Gamma(V_x)$ and $A_\Gamma(W_x)$ are canonically isomorphic. Thus one can speak of A_Γ (isomorphic to $A_\Gamma(V_x)$) and of its commutator ideal C_Γ. Douglas shows that C_Γ is simple, and that if Γ_1 and Γ_2 are subgroups of $\mathbf{R}$, then A_{Γ_1} and A_{Γ_2} are $*$-isomorphic iff Γ_1 and Γ_2 are isomorphic as ordered groups. He obtains other interesting results on these algebras and conjectures that C_{Γ_1} and C_{Γ_2} are $*$-isomorphic implies that Γ_1 and Γ_2 are isomorphic as ordered groups. In this paper we show that C_Γ is an AF-algebra for Γ a subgroup of $\mathbf{Q}$ (the additive rationals) and that in this case we have $K_0(C_\Gamma) = \Gamma$ where $K_0(\ )$ denotes the K_0-group of C_Γ. (For a good account of K_0-theory see Goodearl [3].) From this we deduce that Douglas' conjecture is true for subgroups of $\mathbf{Q}$.

Douglas was led to investigating these algebras A_Γ in the context of a generalized Toeplitz theory. The author has shown they satisfy a certain universal property which can facilitate their analysis, and he has generalized them by associating with every ordered group G a C^*-algebra which reflects both order and algebra properties of G. The results presented here are part of an ongoing investigation of this more general theory, of which the author intends to give a fuller account in a forthcoming paper.

Let H be a separable infinite-dimensional Hilbert space, and let U be the unilateral shift on H. We denote by C the C^*-subalgebra of $B(H)$ generated by U, and by K the commutator ideal of C. (The *commutator ideal* of a

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C^*-algebra B is the closed ideal of B generated by all commutators $ab - ba$ $(a, b \in B)$.) C is the *Toeplitz algebra* and has the following very useful property: If v is an isometry in a unital C^*-algebra B then there exists a unique $*$-homomorphism β from C to B such that $\beta(U) = v$ (Coburn [1]).

We will be retaining the notation U, C and K throughout this paper. Of course, as is well known, K is the ideal of compact operators in $B(H)$, and is thus a simple C^*-algebra.

Let n be a map from the set P of all prime integers into $\mathbf{N} \cup \{\infty\}$ and let $G(n)$ denote the set of all quotients a/b where $a, b \in \mathbf{Z}$, $b > 0$, and if $p \in P$ and p^k divides b then $k \leq n(p)$. This is a subgroup of $\mathbf{Q}$ and in fact every subgroup of $\mathbf{Q}$ is isomorphic to one of these groups $G(n)$ [4, p. 28]. If $n_1, n_2, \ldots$ is a sequence of positive integers such that n_k divides n_{k+1} $(k = 1, 2, \ldots)$ we denote by $\mathbf{Z}(1/n_1, 1/n_2, \ldots)$ the subgroup of $\mathbf{Q}$ generated by $1/n_1, 1/n_2 \ldots$ Using the above mentioned fact, one can show by an elementary argument that every subgroup of $\mathbf{Q}$ is isomorphic to one of this form $\mathbf{Z}(1/n_1, 1/n_2, \ldots)$. This is a crucial point in our analysis of C_Γ.

Proposition 1. *Let $n_1, n_2, \ldots$ be a sequence of positive integers such that n_k divides n_{k+1} $(k = 1, 2, \ldots)$. Define $\phi_k : \mathbf{Z} \to \mathbf{Z}$ by setting $\phi_k(m) = mn_{k+1}/n_k$ $(k = 1, 2, \ldots)$. Then $\mathbf{Z}(1/n_1, 1/n_2, \ldots)$ is the direct limit (in the category of abelian groups) of the sequence of groups and homomorphisms $(\phi_k : \mathbf{Z} \to \mathbf{Z})_{k=1}^{\infty}$.*

Proof. Let Γ denote $\mathbf{Z}(1/n_1, 1/n_2, \ldots)$. Define the homomorphisms $\phi^k : \mathbf{Z} \to \Gamma$ by setting $\phi^k(m) = m/n_k$ $(k = 1, 2, \ldots)$. We have $\phi^{k+1}\phi_k = \phi^k$ $(k = 1, 2, \ldots)$. Suppose that $\psi^k : \mathbf{Z} \to G$ are homomorphisms into an abelian group G such that $\psi^{k+1}\phi_k = \psi^k$ $(k = 1, 2, \ldots)$. Then $\psi^k(1) = \psi^{k+1}\phi_k(1) = \psi^{k+1}(n_{k+1}/n_k) = n_{k+1}/n_k \psi^{k+1}(1)$. It follows that $\psi^k(1) = n_j/n_k \psi^j(1)$ if $k < j$. Thus if $\phi^k(m_1) = \phi^j(m_2)$ then $m_1/n_k = m_2/n_j$ so $\psi^k(m_1) = m_1 n_j/n_k \psi^j(1) = m_2\psi^j(1) = \psi^j(m_2)$. Now since $1/n_1, 1/n_2, \ldots$ generate Γ, Γ is the union of the increasing sequence of subgroups $\phi^1(\mathbf{Z}) \subseteq \phi^2(\mathbf{Z}) \subseteq \cdots$, so we can define a map $\psi : \Gamma \to G$ by setting $\psi(\phi^k(m)) = \psi^k(m)$, and we know that ψ is well defined by the above remarks. It is now clear that ψ is the unique homomorphism γ from Γ to G such that $\gamma\phi^k = \psi^k$ $(k = 1, 2, \ldots)$. Thus we have shown that Γ has the appropriate "universal" or "diagram" property, and so Γ is the direct limit of the sequence $(\phi^k : \mathbf{Z} \to \mathbf{Z})_{k=1}^{\infty}$. $\square$

Remarks. 1. If $\psi : \Gamma_1 \to \Gamma_2$ is an isomorphism of subgroups of $\mathbf{R}$ such that $x \leq y$ iff $\psi(x) \leq \psi(y)$ $(x, y \in \Gamma_1)$ then ψ is called an *order isomorphism*. In this case we have A_{Γ_1} and A_{Γ_2} are $*$-isomorphic.

2. If $\psi : \Gamma_1 \to \Gamma_2$ is an isomorphism of subgroups of $\mathbf{Q}$, then $-\psi$ is one also, and either ψ or $-\psi$ is an order isomorphism. (Proof. If $\Gamma_1 = 0$, there's nothing to prove, so suppose that $x \in \Gamma_1$, $x > 0$. Let $\varepsilon = \psi(x)/|\psi(x)|$. Then $\phi = \varepsilon\psi$ is clearly an isomorphism. If $y \in \Gamma_1$, and $y > 0$, then we can write

$mx = ny$ for some positive integers m and n. We have $m\phi(x) = n\phi(y) \Rightarrow \phi(x)/\phi(y) > 0$ and $\phi(x) > 0$, so therefore $\phi(y) > 0$. Thus $\phi(\Gamma_1^+) \subseteq \Gamma_2^+$. Conversely if $z \in \Gamma_2^+$ then $z = \phi(y)$ for some $y \in \Gamma_1$. If $y < 0$ then $\phi(-y) = -z > 0 \Rightarrow z < 0$, a contradiction. Hence $y \geq 0$, so $\Gamma_2^+ = \phi(\Gamma_1^+)$.) Thus subgroups of **Q** are isomorphic iff they are order isomorphic.

Theorem 2. *Let* Γ *be a subgroup of* **Q**. *Then* C_Γ *is a simple* AF*-algebra and* $K_0(C_\Gamma) = \Gamma$.

Proof. Without loss of generality we may assume that $\Gamma = \mathbf{Z}(1/n_1, 1/n_2, \ldots)$ for some sequence of positive integers $n_1, n_2, \ldots$ such that n_k divides n_{k+1} $(k = 1, 2, \ldots)$. Let $\phi_k : \mathbf{Z} \to \mathbf{Z}$ be defined as before by setting $\phi_k(m) = mn_{k+1}/n_k$. We define $\Psi_k : C \to C$ as the unique $*$-homomorphism such that $\Psi_k(U)$ is U taken to the power of n_{k+1}/n_k and let $\psi_k : K \to K$ be the corresponding restriction of Ψ_k. Likewise we define $\Psi^k : C \to A_\Gamma$ as the unique $*$-homomorphism such that $\Psi^k(U) = V_{1/n_k}$ where $V : \Gamma^+ \to A_\Gamma$ is the one-parameter semigroup of isometries generating A_Γ. We let $\psi^k : K \to C_\Gamma$ be the corresponding restriction. Note that ψ^k is injective since K is simple and $\psi^k \neq 0$ $(\psi^k(1 - UU^*) = 1 - V_{1/n_k}(V_{1/n_k})^* \neq 0$ since V_{1/n_k} is nonunitary). Now if A is the closure of the union of the increasing sequence of C^*-subalgebras of A_Γ, $\Psi^1(C) \subseteq \Psi^2(C) \subseteq \cdots$ then A is a C^*-subalgebra of A_Γ containing the generating set $V_{1/n_k} = \Psi^k(U)$ $(k = 1, 2, \ldots)$, so $A = A_\Gamma$. (To see the sequence is increasing note that $\Psi^k = \Psi^{k+1}\Psi_k$. To see that V_{1/n_k} $(k = 1, 2, \ldots)$ generate A_Γ note that if x is a positive element of Γ then $x = m_1/n_1 + \cdots + m_r/n_r$ where $m_1, \ldots, m_r$ are integers, so $x = ((m_1/n_1)n_r + \cdots + (m_r/n_r)n_r)/n_r = m/n_r$ where m is a positive integer. Thus $V_x = (V_{1/n_r})^m$.) Since $A_\Gamma = A$, it follows that C_Γ is the closure of the union of the increasing sequence of C^*-subalgebras $\psi^k(K)$ $(k = 1, 2, \ldots)$. We now show that C_Γ is the direct limit in the category of C^*-algebras of the sequence of C^*-algebras and $*$-homomorphism $(\psi_k : K \to K)_{k=1}^\infty$ with the $*$-homomorphisms $\psi^k : K \to C_\Gamma$ as "natural" maps. Note that $\psi^{k+1}\psi_k = \psi^k$.

Suppose that $\beta^k : K \to B$ are $*$-homomorphisms into a C^*-algebra B such that $\beta^{k+1}\psi_k = \beta^k$ $(k = 1, 2, \ldots)$. We define the $*$-homomorphism β on the $*$-subalgebra $U\{\psi^k(K) : k = 1, 2, \ldots\}$ by setting $\beta\psi^k(a) = \beta^k(a)$. This is well defined since $\psi^k(a_1) = \psi^k(a_2) \Rightarrow a_1 = a_2$. Necessarily β is norm-decreasing on each C^*-algebra $\psi^k(K)$, and so on their union, which is dense in C_Γ. Thus β extends to a unique $*$-homomorphism $\beta : C_\Gamma \to B$ such that $\beta\psi^k = \beta^k$ $(k = 1, 2, \ldots)$. This means that C_Γ has the appropriate "diagram property" and so C_Γ is the direct limit of the sequence $(\psi_k : K \to K)_{k=1}^\infty$. By general principles of C^*-algebra theory, a direct limit of simple C^*-algebras is simple [5], and a direct limit of AF-algebras is an AF-algebra [3]. Thus since K is a simple AF-algebra, C_Γ is a simple AF-algebra. Also, since the functor K_0 is "continuous" (preserves direct limits), we have $K_0(C_\Gamma)$ is the direct limit in the category of abelian groups of the sequence $(K_0(\psi_k): K_0(K) \to K_0(K))_{k=1}^\infty$. However if $P_k = 1 - U^k(U^k)^*$ then it is well known that $[P_k] = k[P_1]$ and $K_0(K) = \mathbf{Z}[P_1]$, where $[P_k]$ is the "dimension" of the projection P_k in $K_0(K)$. Now $K_0(\psi_k)[P_1] = [\psi_k(P_1)] = [P_{n_{k+1}}/n_k] = n_{k+1}/n_k[P_1]$. It follows that on identifying (as we may) $K_0(K)$ with $\mathbf{Z}$ we see that $K_0(\psi_k)$ is just the homomorphism $\phi_k: \mathbf{Z} \to \mathbf{Z}$. Thus $K_0(C_\Gamma)$ is the direct limit of the sequence $(\phi_k: \mathbf{Z} \to \mathbf{Z})_{k=1}^\infty$. But we saw in Proposition 1 that $\Gamma = \mathbf{Z}(1/n_1, 1/n_2, \ldots)$ is the direct limit of this sequence. Thus $K_0(C_\Gamma)$ is isomorphic to Γ. □

Remark. A_Γ depends on Γ not just as a group, but as an ordered group. However as we saw above, subgroups of $\mathbf{Q}$ are isomorphic iff they are order isomorphic. Thus it follows from Theorem 2 that if Γ_1 and Γ_2 are subgroups of $\mathbf{Q}$ then C_{Γ_1} and C_{Γ_2} are isomorphic iff Γ_1 and Γ_2 are isomorphic groups. Since $\mathbf{Q}$ has infinitely many nonisomorphic subgroups Γ we have infinitely many nonisomorphic C_Γ. By the way (as Douglas pointed out in [2]) C_Γ is not type I if Γ is not isomorphic to $\mathbf{Z}$ (since if C_Γ is type I then C_Γ is isomorphic to $K \Rightarrow C_\Gamma$ is isomorphic to $C_\mathbf{Z}$).

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