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# SIMPLE $C^*$-ALGEBRAS AND SUBGROUPS OF Q

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ABSTRACT. A special case of a conjecture of R. Douglas is solved by an elementary argument using $K_0$-theory.

Let $\Gamma$ be a subgroup of the additive reals $\mathbf{R}$ and let $\Gamma^+ = \{x \in \Gamma : x \geq 0\}$. Douglas [2] defines a *one-parameter semigroup of isometries* to be a homomorphism $x \mapsto V_x$ of $\Gamma^+$ into the set of isometries on some Hilbert space $H$ (i.e. $V_{x+y} = V_x V_y$ for $x, y \in \Gamma^+$ and $V_0 = 1$). Denoting by $A_\Gamma(V_x)$ the $C^*$-algebra generated by all $V_x$ $(x \in \Gamma^+)$ and calling the map $x \mapsto V_x$ *nonunitary* if no $V_x$ is unitary except $V_0 = 1$, he shows that if $x \mapsto V_x$ and $x \mapsto W_x$ are nonunitary one-parameter semigroups of isometries on $\Gamma$ then the algebras $A_\Gamma(V_x)$ and $A_\Gamma(W_x)$ are canonically isomorphic. Thus one can speak of $A_\Gamma$ (isomorphic to $A_\Gamma(V_x)$) and of its commutator ideal $C_\Gamma$. Douglas shows that $C_\Gamma$ is simple, and that if $\Gamma_1$ and $\Gamma_2$ are subgroups of $\mathbf{R}$, then $A_{\Gamma_1}$ and $A_{\Gamma_2}$ are $*$-isomorphic iff $\Gamma_1$ and $\Gamma_2$ are isomorphic as ordered groups. He obtains other interesting results on these algebras and conjectures that $C_{\Gamma_1}$ and $C_{\Gamma_2}$ are $*$-isomorphic implies that $\Gamma_1$ and $\Gamma_2$ are isomorphic as ordered groups. In this paper we show that $C_\Gamma$ is an $AF$-algebra for $\Gamma$ a subgroup of $\mathbf{Q}$ (the additive rationals) and that in this case we have $K_0(C_\Gamma) = \Gamma$ where $K_0(\ )$ denotes the $K_0$-group of $C_\Gamma$. (For a good account of $K_0$-theory see Goodearl [3].) From this we deduce that Douglas' conjecture is true for subgroups of $\mathbf{Q}$.

Douglas was led to investigating these algebras $A_\Gamma$ in the context of a generalized Toeplitz theory. The author has shown they satisfy a certain universal property which can facilitate their analysis, and he has generalized them by associating with every ordered group $G$ a $C^*$-algebra which reflects both order and algebra properties of $G$. The results presented here are part of an ongoing investigation of this more general theory, of which the author intends to give a fuller account in a forthcoming paper.

Let $H$ be a separable infinite-dimensional Hilbert space, and let $U$ be the unilateral shift on $H$. We denote by $C$ the $C^*$-subalgebra of $B(H)$ generated by $U$, and by $K$ the commutator ideal of $C$. (The *commutator ideal* of a

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$C^*$-algebra $B$ is the closed ideal of $B$ generated by all commutators $ab - ba$ $(a, b \in B)$.) $C$ is the *Toeplitz algebra* and has the following very useful property: If $v$ is an isometry in a unital $C^*$-algebra $B$ then there exists a unique $*$-homomorphism $\beta$ from $C$ to $B$ such that $\beta(U) = v$ (Coburn [1]).

We will be retaining the notation $U$, $C$ and $K$ throughout this paper. Of course, as is well known, $K$ is the ideal of compact operators in $B(H)$, and is thus a simple $C^*$-algebra.

Let $n$ be a map from the set $P$ of all prime integers into $\mathbf{N} \cup \{\infty\}$ and let $G(n)$ denote the set of all quotients $a/b$ where $a, b \in \mathbf{Z}$, $b > 0$, and if $p \in P$ and $p^k$ divides $b$ then $k \leq n(p)$. This is a subgroup of $\mathbf{Q}$ and in fact every subgroup of $\mathbf{Q}$ is isomorphic to one of these groups $G(n)$ [4, p. 28]. If $n_1, n_2, \ldots$ is a sequence of positive integers such that $n_k$ divides $n_{k+1}$ $(k = 1, 2, \ldots)$ we denote by $\mathbf{Z}(1/n_1, 1/n_2, \ldots)$ the subgroup of $\mathbf{Q}$ generated by $1/n_1, 1/n_2 \ldots$ Using the above mentioned fact, one can show by an elementary argument that every subgroup of $\mathbf{Q}$ is isomorphic to one of this form $\mathbf{Z}(1/n_1, 1/n_2, \ldots)$. This is a crucial point in our analysis of $C_\Gamma$.

**Proposition 1.** *Let $n_1, n_2, \ldots$ be a sequence of positive integers such that $n_k$ divides $n_{k+1}$ $(k = 1, 2, \ldots)$. Define $\phi_k \colon \mathbf{Z} \to \mathbf{Z}$ by setting $\phi_k(m) = mn_{k+1}/n_k$ $(k = 1, 2, \ldots)$. Then $\mathbf{Z}(1/n_1, 1/n_2, \ldots)$ is the direct limit (in the category of abelian groups) of the sequence of groups and homomorphisms $(\phi_k \colon \mathbf{Z} \to \mathbf{Z})_{k=1}^{\infty}$.*

*Proof.* Let $\Gamma$ denote $\mathbf{Z}(1/n_1, 1/n_2, \ldots)$. Define the homomorphisms $\phi^k \colon \mathbf{Z} \to \Gamma$ by setting $\phi^k(m) = m/n_k$ $(k = 1, 2, \ldots)$. We have $\phi^{k+1}\phi_k = \phi^k$ $(k = 1, 2, \ldots)$. Suppose that $\psi^k \colon \mathbf{Z} \to G$ are homomorphisms into an abelian group $G$ such that $\psi^{k+1}\phi_k = \psi^k$ $(k = 1, 2, \ldots)$. Then $\psi^k(1) = \psi^{k+1}\phi_k(1) = \psi^{k+1}(n_{k+1}/n_k) = n_{k+1}/n_k\psi^{k+1}(1)$. It follows that $\psi^k(1) = n_j/n_k\psi^j(1)$ if $k < j$. Thus if $\phi^k(m_1) = \phi^j(m_2)$ then $m_1/n_k = m_2/n_j$ so $\psi^k(m_1) = m_1 n_j/n_k\psi^j(1) = m_2\psi^j(1) = \psi^j(m_2)$. Now since $1/n_1, 1/n_2, \ldots$ generate $\Gamma$, $\Gamma$ is the union of the increasing sequence of subgroups $\phi^1(\mathbf{Z}) \subseteq \phi^2(\mathbf{Z}) \subseteq \cdots$, so we can define a map $\psi \colon \Gamma \to G$ by setting $\psi(\phi^k(m)) = \psi^k(m)$, and we know that $\psi$ is well defined by the above remarks. It is now clear that $\psi$ is the unique homomorphism $\gamma$ from $\Gamma$ to $G$ such that $\gamma\phi^k = \psi^k$ $(k = 1, 2, \ldots)$. Thus we have shown that $\Gamma$ has the appropriate "universal" or "diagram" property, and so $\Gamma$ is the direct limit of the sequence $(\phi^k \colon \mathbf{Z} \to \mathbf{Z})_{k=1}^{\infty}$. $\square$

*Remarks.* 1. If $\psi \colon \Gamma_1 \to \Gamma_2$ is an isomorphism of subgroups of $\mathbf{R}$ such that $x \leq y$ iff $\psi(x) \leq \psi(y)$ $(x, y \in \Gamma_1)$ then $\psi$ is called an *order isomorphism.* In this case we have $A_{\Gamma_1}$ and $A_{\Gamma_2}$ are $*$-isomorphic.

2. If $\psi \colon \Gamma_1 \to \Gamma_2$ is an isomorphism of subgroups of $\mathbf{Q}$, then $-\psi$ is one also, and either $\psi$ or $-\psi$ is an order isomorphism. (Proof. If $\Gamma_1 = 0$, there's nothing to prove, so suppose that $x \in \Gamma_1$, $x > 0$. Let $\varepsilon = \psi(x)/|\psi(x)|$. Then $\phi = \varepsilon\psi$ is clearly an isomorphism. If $y \in \Gamma_1$, and $y > 0$, then we can write

$mx = ny$ for some positive integers $m$ and $n$. We have $m\phi(x) = n\phi(y) \Rightarrow \phi(x)/\phi(y) > 0$ and $\phi(x) > 0$, so therefore $\phi(y) > 0$. Thus $\phi(\Gamma_1^+) \subseteq \Gamma_2^+$. Conversely if $z \in \Gamma_2^+$ then $z = \phi(y)$ for some $y \in \Gamma_1$. If $y < 0$ then $\phi(-y) = -z > 0 \Rightarrow z < 0$, a contradiction. Hence $y \geq 0$, so $\Gamma_2^+ = \phi(\Gamma_1^+)$.) Thus subgroups of $\mathbf{Q}$ are isomorphic iff they are order isomorphic.

**Theorem 2.** *Let $\Gamma$ be a subgroup of $\mathbf{Q}$. Then $C_\Gamma$ is a simple $AF$-algebra and $K_0(C_\Gamma) = \Gamma$.*

*Proof.* Without loss of generality we may assume that $\Gamma = \mathbf{Z}(1/n_1, 1/n_2, \dots)$ for some sequence of positive integers $n_1, n_2, \dots$ such that $n_k$ divides $n_{k+1}$ $(k = 1, 2, \dots)$. Let $\phi_k: \mathbf{Z} \to \mathbf{Z}$ be defined as before by setting $\phi_k(m) = mn_{k+1}/n_k$. We define $\Psi_k: C \to C$ as the unique $*$-homomorphism such that $\Psi_k(U)$ is $U$ taken to the power of $n_{k+1}/n_k$ and let $\psi_k: K \to K$ be the corresponding restriction of $\Psi_k$. Likewise we define $\Psi^k: C \to A_\Gamma$ as the unique $*$-homomorphism such that $\Psi^k(U) = V_{1/n_k}$ where $V: \Gamma^+ \to A_\Gamma$ is the one-parameter semigroup of isometries generating $A_\Gamma$. We let $\psi^k: K \to C_\Gamma$ be the corresponding restriction. Note that $\psi^k$ is injective since $K$ is simple and $\psi^k \neq 0$ $(\psi^k(1 - UU^*) = 1 - V_{1/n_k}(V_{1/n_k})^* \neq 0$ since $V_{1/n_k}$ is nonunitary). Now if $A$ is the closure of the union of the increasing sequence of $C^*$-subalgebras of $A_\Gamma$, $\Psi^1(C) \subseteq \Psi^2(C) \subseteq \cdots$ then $A$ is a $C^*$-subalgebra of $A_\Gamma$ containing the generating set $V_{1/n_k} = \Psi^k(U)$ $(k = 1, 2, \dots)$, so $A = A_\Gamma$. (To see the sequence is increasing note that $\Psi^k = \Psi^{k+1}\Psi_k$. To see that $V_{1/n_k}$ $(k = 1, 2, \dots)$ generate $A_\Gamma$ note that if $x$ is a positive element of $\Gamma$ then $x = m_1/n_1 + \cdots + m_r/n_r$ where $m_1, \dots, m_r$ are integers, so $x = ((m_1/n_1)n_r + \cdots + (m_r/n_r)n_r)/n_r = m/n_r$ where $m$ is a positive integer. Thus $V_x = (V_{1/n_r})^m$.) Since $A_\Gamma = A$, it follows that $C_\Gamma$ is the closure of the union of the increasing sequence of $C^*$-subalgebras $\psi^k(K)$ $(k = 1, 2, \dots)$. We now show that $C_\Gamma$ is the direct limit in the category of $C^*$-algebras of the sequence of $C^*$-algebras and $*$-homomorphism $(\psi_k: K \to K)_{k=1}^\infty$ with the $*$-homomorphisms $\psi^k: K \to C_\Gamma$ as "natural" maps. Note that $\psi^{k+1}\psi_k = \psi^k$.

Suppose that $\beta^k: K \to B$ are $*$-homomorphisms into a $C^*$-algebra $B$ such that $\beta^{k+1}\psi_k = \beta^k$ $(k = 1, 2, \dots)$. We define the $*$-homomorphism $\beta$ on the $*$-subalgebra $U\{\psi^k(K): k = 1, 2, \dots\}$ by setting $\beta\psi^k(a) = \beta^k(a)$. This is well defined since $\psi^k(a_1) = \psi^k(a_2) \Rightarrow a_1 = a_2$. Necessarily $\beta$ is norm-decreasing on each $C^*$-algebra $\psi^k(K)$, and so on their union, which is dense in $C_\Gamma$. Thus $\beta$ extends to a unique $*$-homomorphism $\beta: C_\Gamma \to B$ such that $\beta\psi^k = \beta^k$ $(k = 1, 2, \dots)$. This means that $C_\Gamma$ has the appropriate "diagram property" and so $C_\Gamma$ is the direct limit of the sequence $(\psi_k: K \to K)_{k=1}^\infty$. By general principles of $C^*$-algebra theory, a direct limit of simple $C^*$-algebras is simple [5], and a direct limit of $AF$-algebras is an $AF$-algebra [3]. Thus since $K$ is a simple $AF$-algebra, $C_\Gamma$ is a simple $AF$-algebra. Also, since the functor $K_0$ is "continuous" (preserves direct limits), we have $K_0(C_\Gamma)$ is the direct limit in the category of abelian groups of the sequence $(K_0(\psi_k): K_0(K) \to K_0(K))_{k=1}^{\infty}$. However if $P_k = 1 - U^k(U^k)^*$ then it is well known that $[P_k] = k[P_1]$ and $K_0(K) = \mathbf{Z}[P_1]$, where $[P_k]$ is the "dimension" of the projection $P_k$ in $K_0(K)$. Now $K_0(\psi_k)[P_1] = [\psi_k(P_1)] = [P_{n_{k+1}}/n_k] = n_{k+1}/n_k[P_1]$. It follows that on identifying (as we may) $K_0(K)$ with $\mathbf{Z}$ we see that $K_0(\psi_k)$ is just the homomorphism $\phi_k: \mathbf{Z} \to \mathbf{Z}$. Thus $K_0(C_\Gamma)$ is the direct limit of the sequence $(\phi_k: \mathbf{Z} \to \mathbf{Z})_{k=1}^{\infty}$. But we saw in Proposition 1 that $\Gamma = \mathbf{Z}(1/n_1, 1/n_2, \ldots)$ is the direct limit of this sequence. Thus $K_0(C_\Gamma)$ is isomorphic to $\Gamma$. $\square$

*Remark.* $A_\Gamma$ depends on $\Gamma$ not just as a group, but as an ordered group. However as we saw above, subgroups of $\mathbf{Q}$ are isomorphic iff they are order isomorphic. Thus it follows from Theorem 2 that if $\Gamma_1$ and $\Gamma_2$ are subgroups of $\mathbf{Q}$ then $C_{\Gamma_1}$ and $C_{\Gamma_2}$ are isomorphic iff $\Gamma_1$ and $\Gamma_2$ are isomorphic groups. Since $\mathbf{Q}$ has infinitely many nonisomorphic subgroups $\Gamma$ we have infinitely many nonisomorphic $C_\Gamma$. By the way (as Douglas pointed out in [2]) $C_\Gamma$ is not type I if $\Gamma$ is not isomorphic to $\mathbf{Z}$ (since if $C_\Gamma$ is type I then $C_\Gamma$ is isomorphic to $K \Rightarrow C_\Gamma$ is isomorphic to $C_\mathbf{Z}$).

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