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REMARK ON WALTER'S INEQUALITY FOR SCHUR MULTIPLIERS

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ABSTRACT. We extend and give another proof of Walter's inequality: For a linear operator $T \in L(l^2(X))$

$$\|T\|^2_{V_2(X)} \leq \||T|_l\|_{V_2(X)}\||T|_r\|_{V_2(X)},$$

where $V_2(X)$ is the Banach algebra of Schur multipliers on $L(l^2(X))$ and $|T|_l = (TT)^{*1/2}$, $|T|_r = |T^*|_l$.

1. INTRODUCTION

Let X be a set with counting measure. The algebra of all bounded operators on $l^2(X)$, denoted $L(l^2(X))$, can be identified with the functions $k: X \times X \to C$, called also kernels, in the following way: For $T \in L(l^2(X))$ and $x, y \in X$, $k(x,y) = \langle T\delta_x, \delta_y\rangle$.

After C. Herz [H] and G. Bennett [Be], we say that a kernel $a(x,y)$ is a *Schur multiplier* of $L(l^2(X))$ if for every $k \in L(l^2(X))$, $a \circ k \in L(l^2(X))$ and $\|a\|_{V_2(X)} = \sup\{\|a \cdot k\| : \|k\| = 1\}$ is finite. Here the multiplication $a \circ k$ is the pointwise product of kernels called also the *Schur product* i.e. $(a \circ k)(x,y) = a(x,y)k(x,y)$ and the norm $\| \ \|$ is the operator norm on $l^2(X)$.

It was shown by M. Walter [W] that if X is a finite set, then for a kernel $a \in L(l^2(X))$ we have the following inequality:

$$(**) \qquad \|a\|^2_{V_2(X)} < \||a|_l\|_{V_2(X)}\||a|_r\|_{V_2(X)},$$

where $|a|_l$ is the left modulus of an operator a defined as follows: $|a|_l = (aa^*)^{1/2}$ and the right modulus $|a|_r = |a^*|_l$. In this note we give another and rather natural proof of Walter's result and we even prove that the inequality $(**)$ holds for all bounded operators for an arbitrary set X.

2. WALTER'S INEQUALITY

Let us start with the following Lemma, which is probably well known, but for completeness we include a proof.

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Lemma. (i) *If* $b(x,y) = \sum_z A(x,z)B(z,y)$, *where*

$$B\colon l^1(X) \longrightarrow l^2(X) \quad and \quad A\colon l^2(X) \longrightarrow l^\infty(X),$$

then

$$\|b\|_{V_2(X)} \le \|B\|_{1,2}\|A\|_{2,\infty}.$$

(ii) *If the kernel* $a(x,y)$ *is positive definite and bounded on* $X \times X$, *then* $\|a\|_{V_2(X)} = \sup_{x\in X} a(x,x)$.

(iii) *If* $a \in L(l^2(X))$, *then*

$$\|a^*a\|_{V_2(X)} = \|a\|_{1,2}^2, \quad \|aa^*\|_{V_2(X)} = \|a\|_{2,\infty}^2.$$

Proof. (i) From the definition of V_2-norm we have

$$\|b\|_{V_2(X)} = \sup_{\|k\|=1}\left\{\left|\sum_{x,y} b(x,y)k(x,y)u(x)v(y)\right| : |u|_2 = \right.$$

$$\left. = |v|_2 = 1, u \text{ and } v \text{ have finite supports}\right\}.$$

Since functions u and v have finite supports, therefore we can interchange the summation and by the Cauchy-Schwarz inequality we get

$$\begin{aligned}\left|\sum_{x,y} b(x,y)u(x)v(y)\right| &= \left|\sum_z\sum_{x,y} k(x,y)A(x,z)u(x)B(z,y)v(y)\right| \\ &\le \|k\|\sum_z\left(\sum_x |A(x,z)u(x)|^2\right)^{1/2}\left(\sum_y |B(z,y)v(y)|^2\right)^{1/2} \\ &\le \|k\|\left(\sum_{z,x} |A(x,z)u(x)|^2\right)^{1/2}\left(\sum_{z,y} |B(z,y)v(y)|^2\right)^{1/2} \\ &\le \|k\|\|A\|_{2,\infty}\|B\|_{1,2}|u|_2|v|_2.\end{aligned}$$

The last inequality follows from the well-known fact (see for example [Be]) that

$$\|A\|_{2,\infty} = \sup_x\left(\sum_z |A(x,z)|^2\right)^{1/2} \quad \text{and} \quad \|B\|_{1,2} = \sup_y\left(\sum_z |B(z,y)|^2\right)^{1/2}.$$

Proof of (ii). Since $\|a\|_{V_2(X)} \ge \sup_{x,y}|a(x,y)|$ always and the kernel a is positive and bounded, by the Cauchy-Schwarz inequality we have therefore

$$|a(x,y)|^2 \le a(x,x)a(y,y),$$

which implies that

$$\sup_{x,y}|a(x,y)| = \sup_x |a(x,x)|.$$

On the other hand since

$$a(x,y) = \langle\delta_x, \delta_y\rangle_a, \qquad x, y \in X,$$

where the scalar product $\langle\ ,\ \rangle_a$ is induced by the kernel a.

Now we can apply part (i) of our Lemma and we get

$$\|a\|_{V_2(X)} \leq \sup_x \langle\delta_x , \delta_x\rangle_a = \sup_x a(x , x).$$

Hence $\|a\|_{V_2(X)} = \sup_x a(x , x)$.

Proof of (iii). Since a^*a is a positive definite operator, by (ii) we obtain

$$\|a^* a\|_{V_2(X)} = \sup_x \sum_z a^*(x , z)a(z , x) = \|a\|_{1,2}^2.$$

By the duality argument we have $\|aa^*\|_{V_2(X)} = \|a\|_{2,\infty}^2$.

We are now ready to prove the extension of M. Walter's inequality.

Theorem. *If a is a bonded linear operator on $l^2(X)$, then*

(i) $\|a\|_{V_2(X)}^2 \leq \||a|_r\|_{V_2(X)}\||a|_l\|_{V_2(X)}$.

(ii) *For $0 \leq t \leq 2$ we have*

$$\|a\|_{V_2(X)}^2 \leq \||a|_r^t\|_{V_2(X)}\||a|_l^{(2-t)}\|_{V_2(X)}.$$

Proof. First we give a slightly different interpretation of the Lemma. We showed that if a matrix a is a Schur multiplier of the form $a = AB$, where $B\colon l_1(X) \to l_2(X)$ and $A\colon l_2(X) \to l_\infty(X)$, then

$$\|a\|_{V_2(X)} \leq \|A\|_{2,\infty}\|B\|_{1,2}.$$

We shall prove (ii) of our Theorem using the polar decomposition of the operator a and finding a quite good factorization. It is well known (see [GK] that $a = u|a|_r = |a|_l u$, where u is a suitable partial isometry operator on $l_2(X)$, and $|a|_r = (a^*a)^{1/2}$ and $|a|_l = (aa^*)^{1/2}$.

Hence one can show that in our case for $t > 0$ we have $u|a|_r^t = |a|_l^t u$. Therefore

$$a = u|a|_r = |a|_l^{(1-t/2)} u|a|_r^{t/2}.$$

Since $\|u\|_{2,2} \leq 1$, by the Lemma we get

$$\begin{aligned}
\|a\|_{V_2(X)}^2 &\leq \|u|a|_r^{t/2}\|_{1,2}^2\||a|_l^{(1-t/2)}\|_{2,\infty}^2 \\
&\leq \|u\|_{2,2}^2\||a|_r^{t/2}\|_{1,2}^2\||a|_l^{(2-t)}\|_{V_2(X)} \\
&\leq \||a|_r^t\|_{V_2(X)}\||a|_l^{(2-t)}\|_{V_2(X)}.
\end{aligned}$$

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