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AN ITERATION PROCESS FOR NONEXPANSIVE MAPPINGS WITH APPLICATIONS TO FIXED POINT THEORY IN PRODUCT SPACES

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ABSTRACT. A uniform transfinite iteration procedure for selecting fixed points of nonexpansive mappings is introduced. This procedure, which applies to arbitrary nonexpansive mappings in Banach spaces having Kadec-Klee norm and to strictly contractive mappings in reflexive Banach spaces, is used to generalize a fixed point theorem of Kirk and Sternfeld for nonexpansive mappings in product spaces.

1. INTRODUCTION

Let E be a Banach space, $K \subset E$, and $T: K \to K$ nonexpansive $(\|T(u) - T(v)\| \le \|u - v\|,\ u,\ v \in K)$. It is known [1] that if E is uniformly convex and K closed and convex, then the mapping $I - T$ is demiclosed on K in the sense: if $\{u_j\}$ is a sequence in K which converges weakly to u and if $\{(I - T)(u_j)\}$ converges strongly to w, then $u \in K$ and $(I - T)(u) = w$. A procedure, which can be viewed as an extension of this fact, is introduced below and, in turn, applied to generalize a result of Kirk-Sternfeld [6]. This generalized result extends the original by replacing the uniform convexity assumption with an assumption even weaker than "nearly uniformly convex".

2. PRELIMINARIES

We use $B(x;r)$ to denote the closed ball centered at $x \in E$ with radius $r \ni 0$, and $\overline{\text{conv}}(S)$ to denote the closed convex hull of $S \subset E$.

Definition. The norm of a Banach space E is said to be *Kadec-Klee* (KK) if for $\{x_n\}$ in $B(0;1)$,

$$\left.\begin{array}{l}\text{weak-lim } x_n = x \\ \text{sep}(x_n) \ni 0\end{array}\right\} \Rightarrow \|x\| < 1.$$

Equivalently, the norm of E is (KK) if whenever $\{x_n\}$ converges weakly but not strongly to $x \in E$, $\|x\| < \liminf \|x_n\|$.

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We formulate the principal results in this paper in spaces which are reflexive and have (KK) norm. Thus they apply to spaces studied by Huff [3] which are more general than uniformly convex spaces. We need the following fact which is essentially a consequence of the Eberlein-Smulian Theorem. In particular, it follows trivially from results of [8].

Proposition 1. *Suppose E is a reflexive Banach space with (KK) norm. Let $\{x_\alpha : \alpha \in \mathbf{A}\}$ be a net in $B(0;1)$ which converges weakly but not strongly to x. Then $\|x\| < 1$.*

3. Iteration procedure

In defining the procedure below we utilize the concept of a universal (or ultra) net. A net $\{x_\alpha\}$ in a set S is said to be *universal* (see Kelley [5]) if for each subset U of S, either $\{x_\alpha\}$ is eventually in U or $\{x_\alpha\}$ is eventually in the complement of U. The following facts are pertinent to our application of this concept (see [5, p. 81]).

(a) Every net in a set has a universal subnet.

(b) If $f: S_1 \to S_2$ is any mapping, and if $\{x_\alpha\}$ is a universal net in S_1, then $\{f(x_\alpha)\}$ is a universal net in S_2.

(c) If S is compact and if $\{x_\alpha\}$ is a universal net in S, then $\lim_\alpha x_\alpha$ exists.

We associate with each $\alpha \in \Omega_0$ (the set of all countable ordinals) a *fixed* universal subnet $\{\beta_{\mu(\alpha)} : \mu(\alpha) \in M_\alpha\}$ of α. (Specifically, M_α is a directed set with $\varphi_\alpha : M_\alpha \to \{\beta \in \Omega_0 : \beta < \alpha\}$ isotone and cofinal. Denote: $\varphi_\alpha(\mu(\alpha)) = \beta_{\mu(\alpha)}$. Thus if $\mu_1(\alpha) \le \mu_2(\alpha)$ in M_α then $\beta_{\mu_1(\alpha)} \le \beta_{\mu_2(\alpha)}$ and given $\beta < \alpha$ there exists $\mu(\alpha) \in M_\alpha$ such that $\beta_{\mu(\alpha)} > \beta$.)

Now let K be a weakly compact convex subset of a Banach space and $f: K \to K$. Fix $x_0 \in K$, let $\xi \in \Omega_0$, and make the inductive assumption $\{x_\alpha : \alpha < \xi\} \subset K$ has been defined. Set:

(1) $x_\xi = f(x_{\xi'})$ if $\xi = \xi' + 1$;

(2) $x_\xi = \text{weak-}\lim_{\mu(\xi)} x_{\beta_{\mu(\xi)}}$ if ξ is a limit ordinal.

Clearly (1) and (2) define a net $\{x_\alpha : \alpha \in \Omega_0\}$ in K with initial point x_0.

In the following theorem we apply the above procedure to the study of nonexpansive mappings. It will be important to note for later purposes that this procedure is independent of the mapping f (in the sense that the same indices are always involved in the limiting steps).

Theorem 3.1. *Let E be a reflexive Banach space with (KK) norm and let K be a closed and convex subset of E. Suppose $T: K \to K$ is nonexpansive with nonempty fixed point set P. Set $f = (I+T)/2$. Then for each $x_0 \in K$, the net $\{x_\alpha : \alpha \in \Omega_0\}$ as defined in (1)-(2) is eventually in P (and hence constant).*

Proof. First note that P is also the fixed point set of f in K. Fix $p \in P$. Since T, hence f, is nonexpansive and since the norm of E is weakly lower semicontinuous, the net $\{\|x_\alpha - p\| : \alpha \in \Omega_0\}$ is nonincreasing. Also, a result of Ishikawa [4] implies $\lim \|f^n(x_\alpha) - f^{n+1}(x_\alpha)\| = 0$, $\alpha \in \Omega_0$. Now suppose

$\alpha \in \Omega_0$ is given and let $\alpha' = \alpha + \omega$ (ω denotes the ordinal associated with $\mathbf{N}$). By definition $x_{\alpha'} =$ weak-$\lim_{\mu(\alpha')} x_{\beta_{\mu(\alpha')}}$ where $\{x_{\beta_{\mu(\alpha')}} : \mu(\alpha') \in M_{\alpha'}\}$ is a universal subnet of $\{x_\beta : \beta < \alpha'\}$. It follows that $x_{\alpha'} =$ weak-$\lim_\mu f^{n_\mu}(x_\alpha)$ where $\{n_\mu\}$ is a (universal) subnet of ω. Since $\lim_\mu \|f^{n_\mu}(x_\alpha) - f^{n_\mu+1}(x_\alpha)\| = 0$, if $\{f^{n_\mu}(x_\alpha)\}$ converges strongly to $x_{\alpha'}$, then $x_{\alpha'} \in P$. Otherwise, Proposition 1 implies

$$(*) \qquad \|x_{\alpha'} - p\| < \lim_\mu \|f^{n_\mu}(x_\alpha) - p\| \leq \|x_\alpha - p\|.$$

Since Ω_0 is uncountable, it follows that $x_\alpha = p$ for some $\alpha \in \Omega_0$. This completes the proof.

A mapping $f: K \to K$ is said to be *contractive* if $\|f(u) - f(v)\| < \|u - v\|$ for $u, v \in K$, $u \neq v$. A minor modification of the above yields:

Theorem 3.2. *Let K be a weakly compact subset of a Banach space E, and suppose $f: K \to K$ is contractive with (unique) fixed point $p \in K$. Then for each $x_0 \in K$ the net $\{x_\alpha : \alpha \in \Omega_0\}$ as defined in (1)–(2) is eventually constant and equal to p.*

Proof. The proof is identical with the above except that $(*)$ is established by invoking the fact f is contractive. Thus, if $x_\alpha \neq p$ for $\alpha \in \Omega_0$,

$$\|f^{n_\mu+1}(x_\alpha) - p\| < \|f^{n_\mu}(x_\alpha) - p\|$$

from which $\|x_{\alpha'} - p\| < \|x_\alpha - p\|$.

4. Applications

Suppose $(E, \| \|_E)$ and $(F, \| \|_F)$ are Banach spaces and let $E \oplus F$ denote the product space with norm

$$\|(x, y)\| = \max\{\|x\|_E, \|y\|_F\}, \qquad x \in E, y \in F.$$

It was shown in Kirk-Sternfeld [6] that for E uniformly convex, if $X \subset E$ is bounded closed and convex and $Y \subset F$ bounded closed and separable, then the assumption that Y has the fixed point property for nonexpansive mappings assures that the same is true of $X \oplus Y$. It was proved in [7] that the separability assumption on Y can be removed. Here we apply Theorem 3.1 to generalize this result further.

Theorem 4.1. *Let E and F be Banach spaces and suppose E has* (KK) *norm. Let $X \subset E$ and $Y \subset F$. Suppose X is weakly compact and convex, and suppose both X and Y have the fixed point property for nonexpansive mappings. Then $X \oplus Y$ has the fixed point property for nonexpansive mappings.*

Proof. Suppose $T: X \oplus Y \to X \oplus Y$ is nonexpansive. Let P_i, $i = 1, 2$, denote, respectively, the coordinate projections of $E \oplus F$ onto E and F, and for fixed $y \in Y$ define $T_y : X \to X$ by

$$T_y(x) = P_1 \circ T(x, y), \qquad x \in X.$$

Set $S_y = (I + T_y)/2$, fix $x_0 \in X$, and let $\{x_{\alpha,y}\}$ be the iteration process defined by (1)–(2) of Section 2 taking $f = S_y$. Since T_y, hence S_y, is nonexpansive, by assumption S_y has a nonempty fixed point set P in X. By Theorem 3.1, $x_{\alpha,y} \equiv y(1) \in P$ for all $\alpha \in \Omega_0$ sufficiently large. Thus $P_1 \circ R(y(1), y) \equiv y(1)$.

Now let $u, v \in Y$. Then

$$\begin{aligned}\|S_u(x_0) - S_v(x_0)\|_E &= \tfrac{1}{2}\|T_u(x_0) - T_v(x_0)\|_E \\ &\leq \tfrac{1}{2}\|T(x_0, u) - T(x_0, v)\| \leq \|u - v\|_F.\end{aligned}$$

We make the inductive assumption that $\|x_{\beta,u} - x_{\beta,v}\|_E \leq \|u - v\|_F$ for all $\beta < \alpha \in \Omega_0$. If α is a limit ordinal, then $\|x_{\alpha,u} - x_{\alpha,v}\|_E \leq \|u - v\|_F$ by weak lower semicontinuity of the norm. If $\alpha = \alpha' + 1$, then

$$\begin{aligned}&\|x_{\alpha,u} - x_{\alpha,v}\|_E \\ &\quad = \|S_u(x_{\alpha',u}) - S_v(x_{\alpha',v})\|_E \\ &\quad \leq \tfrac{1}{2}\|T_u(x_{\alpha',u}) - T_v(x_{\alpha',v})\|_E + \tfrac{1}{2}\|x_{\alpha',u} - x_{\alpha',v}\|_E \\ &\quad = \tfrac{1}{2}\|P_1 \circ T(x_{\alpha',u}, u) - P_1 \circ T(x_{\alpha',v}, v)\|_E + \tfrac{1}{2}\|x_{\alpha',u} - x_{\alpha',v}\|_E \\ &\quad \leq \tfrac{1}{2}\|T(x_{\alpha',u}, u) - T(x_{\alpha',v})\|_E + \tfrac{1}{2}\|x_{\alpha',u} - x_{\alpha',v}\|_E \\ &\quad \leq \tfrac{1}{2}\|(x_{\alpha',u}, u) - (x_{\alpha',v}, v)\| + \tfrac{1}{2}\|x_{\alpha',u} - x_{\alpha',v}\|_E \\ &\quad = \tfrac{1}{2}\max\{\|x_{\alpha',u} - x_{\alpha',v}\|_E, \|u - v\|_F\} + \tfrac{1}{2}\|x_{\alpha',u} - x_{\alpha',v}\|_E \\ &\quad \leq \|u - v\|_F.\end{aligned}$$

This completes the induction, yielding

$$\|x_{\alpha,u} - x_{\alpha,v}\|_E \leq \|u - v\|_F, \qquad \alpha \in \Omega_0, u, v \in Y.$$

It follows that

$$\|u(1) - v(1)\|_E \leq \|u - v\|_F.$$

Now let $g: Y \to Y$ be defined by

$$g(y) = P_2 \circ T(y(1), y), \qquad y \in Y.$$

Then for $u, v \in Y$,

$$\begin{aligned}\|g(u) - g(v)\|_F &= \|P_2 \circ T(u(1), u) - P_2 \circ T(v(1), v)\|_F \\ &\leq \|T(u(1), u) - T(v(1), v)\| \\ &\leq \max\{\|u(1) - v(1)\|_E, \|u - v\|_F\} \\ &= \|u - v\|_F.\end{aligned}$$

Therefore g is nonexpansive on Y and thus has a fixed point $y \in Y$; hence $(y(1), y)$ is a fixed point of T in $X \oplus Y$.

By applying Theorem 3.2 instead of Theorem 3.1 the above argument also yields:

Theorem 4.2. *Let E and F be Banach spaces with $X \subset E$ and $Y \subset F$. Suppose X is weakly compact and suppose both X and Y have the fixed point property for contractive mappings. Then $X \oplus Y$ has the fixed point property for contractive mappings.*

Proof. Assume the notation of the proof of Theorem 4.1 with $T: X\oplus Y \to X\oplus Y$ contractive. Fix $x_0 \in X$. For $y \in Y$ let $\{x_{\alpha,y}\}$ denote the iteration process (starting at x_0) defined by (1)–(2) taking $f = T_y$. Since T_y is contractive, $x_{\alpha,y} = y(1) = T_y(y(1))$ for $\alpha \in \Omega_0$ sufficiently large. Also, as above,

$$\|x_{\alpha,u} - x_{\alpha,v}\|_E \le \|u - v\|_F, \qquad \alpha \in \Omega_0, u, v \in Y,$$

from which

$$\|u(1) - v(1)\|_E \le \|u - v\|_F.$$

Thus, if $u \neq v$,

$$\begin{aligned} \|g(u) - g(v)\|_F &= \|P_2 \circ T(u(1), u) - P_2 \circ T(v(1), v)\|_E \\ &\le \|T(u(1), u) - T(v(1), v)\| \\ &< \|(u(1), u) - (v(1), v)\| \\ &= \max\{\|u(1) - v(1)\|_E, \|u - v\|_F\} \\ &= \|u - v\|_F. \end{aligned}$$

Therefore g is contractive on Y and thus has a fixed point $y \in Y$, completing the proof.

Added in proof. T. Kuczumow [Fixed point theorems in product spaces] has recently generalized Theorem 4.1 by replacing the assumption that E has (KK) norm with the assumption that every nonexpansive $f: X \to X$ has a fixed point in each nonempty closed convex f-invariant subset of X. Kuczumow uses Tychonoff's Theorem and a method of R. E. Bruck.

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