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GRADED RINGS AND KRULL ORDERS

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ABSTRACT. Let R be a faithfully S-graded ring, where S is a submoniod of a torsion-free commutative group and S has no nontrivial units. In case R is a prime Krull order we give necessary and sufficient conditions for R to be a crossed product (respectively a polynomial ring).

1. INTRODUCTION

We fix some notation and terminology. All rings are associative with unity and all monoids S are torsion-free cancellative commutative, that is all monoids are contained in a torsion-free commutative group. By e we denote the identity of S, and $\langle S\rangle$ denotes the group of quotients of S (we use the multiplicative notation). The group of units of S is denoted $U(S)$.

A ring R is said to be S-graded, where S is a monoid, if $R = \bigoplus_{s\in S} R_s$, a direct sum of Abelian groups, and $R_sR_t \subseteq R_{st}$ for all s, $t \in S$. The set $h(R) = \bigcup_{s\in S} R_s$ is the set of homogeneous elements of R. We denote by C the set of homogeneous elements that are regular in R, and $C_s = C \cap R_s$. We say R is faithfully graded [6] if $\underline{l}(R_s) = \underline{r}(R_s) = 0$ for all $s \in S$. Here $\underline{l}(R_s)$ (respectively $\underline{r}(R_s)$) denotes the left (respectively right) annihilator of R_s in R. If, moreover, S is a group and $R_sR_t = R_{st}$ then R is said to be strongly graded. Clearly strongly graded rings are faithfully graded. If A is a subset of S, then $R_{[A]} = \bigoplus_{a\in A} R_a$. A left or right ideal I of R is homogeneous if $I = \bigoplus_{s\in S} R_s \cap I$.

Throughout, by the term Goldie ring is meant left and right Goldie ring.

Lemma 1.1. *Let R be an S-graded ring. If R is a prime Goldie ring such that every essential homogeneous left (or right) ideal of R intersects C, then C is a left and right Ore set of R.*

Proof. We only prove that the multiplicatively closed set C is a left Ore set. Let $r \in R_t$ and $c \in C_s$, s, $t \in S$. Then $(Rc :_l r) = \{x \in R | xr \in Rc\}$ is a

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homogeneous left ideal of R. Since R is a prime left Goldie ring $(Rc :_l r)$ is also essential. Hence by the assumption $(Rc :_l r) \cap C \neq \emptyset$, and thus $c'r = r'c$ for some $c' \in C$ and $r' \in h(R)$. The left Ore condition for a general r follows easily. □

Note that the condition of the lemma holds for $P.I.$-rings, but in general it is not satisfied (cf. [11]).

If R is an S-graded prime Goldie ring such that every essential one-sided ideal intersects C, then we denote by Q^g the graded classical ring of quotients, i.e., $Q^g = C^{-1}R = \{c^{-1}r | r \in R,\ c \in C\} = \{rc^{-1} | r \in R,\ c \in C\} = RC^{-1}$. Clearly Q^g is a $\langle S \rangle$-graded ring.

Let R be a Krull order in the sense of Chamarie [3, 4], i.e. R is a prime Goldie ring which is a left and right Krull order. It follows that R is a maximal order with the ascending chain condition on integral c-ideals. An R-ideal I is called a c-ideal if $I = (R : (R : I))$, where $(R : I) = \{q \in Q_{cl}(R) | qI \subset R\} = \{q \in Q_{cl}(R) | Iq \subseteq R\}$. By $G(R)$ we denote the free Abelian group of c-ideals. Let $\mathbf{P}^c(R)$ (respectively, $\mathbf{P}^n(R)$) be the group all principal c-ideals I of R generated by a central (respectively normalizing) element. This means $I = Rc = cR$ (respectively $I = Rn = nR$) where $0 \neq c$ belongs to the quotient field of the center $Z(R)$ of R (respectively $0 \neq n \in Q_{cl}(R)$, the classical ring of quotients of R). The normalizing (respectively central) class group $Cl^n(R)$ (respectively $Cl^c(R)$) of R is the quotient group $G(R)/\mathbf{P}^n(R)$ (respectively $G(R)/\mathbf{P}^c(R)$). If moreover R is S-graded, then we denote by $Cl^n_g(R) = G_g(R)/\mathbf{P}^n_g(R)$ the graded normalizing class group, and by $Cl^c_g(R) = G_g(R)/\mathbf{P}^c_g(R)$ the graded central class group; where $G_g(R)$ is the subgroup of all homogeneous c-ideals of $G(R)$, and $\mathbf{P}^n_g(R)$ and $\mathbf{P}^c_g(R)$ are defined as above but with n and c homogeneous elements of Q^g, respectively $Q^g(Z(R))$.

In recent years several papers on Krull orders and graded rings have appeared (for example [2, Proposition 5.11; 3, Proposition 3.3; 9, Corollary 4.7; 10, Corollary 2.5; 11, Corollary 3.9; 12, Theorem 3.15 and Corollary 3.17]). The attention was focused on proving that certain graded rings R, such as for example crossed products $R_e * S = R_e[S, \sigma, \gamma]$ (σ a collection of automorphisms, γ a 2-cocycle) over a (semi-) group S with ascending chain condition on cyclic subgroups, are Krull orders whenever R_e is a Krull order. Note that a S-graded ring R, S a monoid, is called a crossed product if for every $s \in S$ there exists a regular element $\bar{s}$ in R_s such that $R_s = R_e\bar{s} = \bar{s}R_e$ and $R_sR_t = R_{st}$ for all s, $t \in S$. Or equivalently, $R = R_e[S, \sigma, \gamma]$ (the notations and definition are the same as in the group case), where σ is a collection of automorphisms of R_e and γ is a 2-cocycle. In this paper we deal with the converse problem: when is a S-graded Krull order R a crossed product? Recently the second author [13] gave an answer in the commutative case if R is a unique factorization domain. In this paper we extend the latter result to the noncommutative case. However our method is completely different from the one used in [13].

We will prove the following:

Theorem 1.2. *Let S be a torsion-free cancellative commutative monoid with $U(S) = \{e\}$. Let R be a faithfully S-graded ring. If R is a Krull order with $Cl_g^n(R) = \{1\}$ then the following are equivalent:*

(1) *Every essential homogeneous left (or right) ideal of R intersects C, and $Q_e^g \subseteq C_e^{-1} R_e \cap R_e C_e^{-1}$;*

(2) *$R \cong R_e * S$, a crossed product, and the isomorphism preserves the grading.*

If any of these conditions holds then S is a free Abelian monoid. If moreover $Cl_g^c(R) = \{1\}$ then $R \cong R_e[X_i | i \in I]$, a polynomial ring in commuting variables, and $|I|$ is the torsion-free rank of S.

The proof of (2) implies (1) is easy. Since R is prime Goldie it follows that R_e is semiprime Goldie (see for example [8]). It is then well known (see for example [9]) that $Q_e^g = Q_{cl}(R_e) = C_e^{-1} R_e = R_e C_e^{-1}$.

Note that if $U(S)$ is not trivial then the theorem does not hold anymore. The example on p. 88 in [1] gives a group graded unique factorization domain R satisfying condition 1, but not condition 2. Indeed, with notations as in [1], let $R = A_{\mathbf{Q}} = K[X_q | q \in \mathbf{Q}]$. Because $\mathbf{Q}$ is a group one easily verifies that condition 1 holds. However if we assume that $R = R_0 * \mathbf{Q}$, a crossed product, then every R_q, $0 \neq q \in \mathbf{Q}$, contains a unit. However, since R is also a polynomial ring, the only units of R are in K and thus in R_0; a contradiction. Hence R is not a crossed product. So for general semigroups S we only can apply Theorem 1.2 for the natural $S/U(S)$ gradations (where the quotient semigroup is defined just as in the group case). In case $S = U(S)$ the latter results however have no meaning.

In the remainder of the paper we prove (1) implies (2).

2. Proof of the theorem

Let R be as in the statement of the theorem, and assume condition (1) is satisfied. Hence, by Lemma 1.2, C is a left and right Ore set in R. Under these assumptions we obtain

Lemma 2.1. *Q^g is strongly $\langle S\rangle$-graded with $Q_{[S]}^g = C_e^{-1} R = R C_e^{-1}$. In particular $Q_{[S]}^g$ is a left and right localization of R.*

Proof. Let $s \in S$. Since $\underline{r}(RR_s) = 0$ and because R is a prime Goldie ring, RR_s is an essential homogeneous left ideal of R. So there exists $c \in C_{as} \cap R_a R_s$ for some $a \in S$. It follows that $0 \neq c^{-1} R_a \subseteq Q_{s^{-1}}^g$, and thus $Q_{s^{-1}}^g Q_s^g = Q_e^g$. Replacing RR_s by $R_s R$, $Q_s^g Q_{s^{-1}}^g = Q_e^g$ follows similarly. Because an arbitrary element x of $\langle S\rangle$ is of the form $s^{-1}t$, for some s, $t \in S$, it is clear that $Q_x^g Q_{x^{-1}}^g = Q_e^g$. Hence Q^g is strongly $\langle S\rangle$-graded.

It is now clear that all regular elements of R_e are regular in Q^g. Thus $C_e = \{c_e \in R_e | c_e \text{ regular in } R_e\}$. Hence all elements of C_e are invertible in Q^g and thus in Q_e^g. Since $\langle S\rangle$ is a torsion-free commutative group it is well known that $\langle S\rangle$ can be ordered. Hence it follows from [8] that Q_e^g is semiprime and hence (cf. [11]) Q_e^g is a semiprime Goldie ring. Now, as $Q_e^g \subseteq C_e^{-1}R_e \cap R_e C_e^{-1}$ we obtain then that Q_e^g is its own classical ring of quotients, in particular Q_e^g has no nontrivial essential one-sided ideals. As $Q_{s^{-1}}^g R_s$ is such a left ideal we obtain $Q_{s^{-1}}^g R_s = Q_e^g$; and similarly $Q_e^g = R_s Q_{s^{-1}}^g$ for every $s \in S$. Hence $Q_s^g = Q_s^g(Q_{s^{-1}}^g R_s) = Q_e^g R_s = C_e^{-1} R_s$ and similarly $Q_s^g = R_s C_e^{-1}$ for every $s \in S$. □

The semigroup S is called a Krull semigroup [5] if S is a maximal order (i.e. $(A : A) = \{x \in \langle S\rangle | xA \subseteq A\} = S$ for every nonempty ideal A of S) and S satisfies the ascending chain condition on integral c-ideals (also called divisorial ideals). The definition of a c-ideal is similar as in the ring case (see [7, p. 215]). Further a c-ideal A is called integral if $A \subseteq S$. If moreover S has trivial class group, or equivalently each nonunit of S has a unique factorization into irreducible elements of S, then S is called factorial. In [7] it is shown that S is factorial if and only if $S \cong U(S) \times S'$ is a direct product, where S' is a free Abelian monoid.

Again let R be as in the statement of the theorem, and assume condition (1) is satisfied. We obtain:

Lemma 2.2. *$Q_{[S]}^g$ is a Krull order with $Cl_g^n(Q_{[S]}^g) = \{1\}$, and S is a factorial monoid.*

Proof. Put $B = Q_{[S]}^g$. Since B is a left and right localization of R it follows from [3] that, (1) B is a Krull order, and (2) for every $I \in G_g(B)$, $I \cap R \in G_g(R)$. Hence $Cl_g^n(R) = \{1\}$ implies $Cl_g^n(B) = \{1\}$.

To show that S is a maximal order, let $x \in \langle S\rangle$ and let A be a nonempty ideal of S with $xA \subseteq A$. Hence $Q_x^g B_{[A]} \subseteq B_{[A]}$. As $B_{[A]}$ is a B-ideal of the Krull order B, we obtain $Q_x^g \subseteq B$. In particular $x \in S$.

Let $I = \bigoplus_{a\in A} I_a$, A a S-ideal (also called fractional ideal) of S, be an arbitrary homogeneous B-ideal. Since Q^g is strongly $\langle S\rangle$-graded with $Q_{[S]}^g = B$, and since Q_e^g has no nontrivial dense ideals, one easily verifies that $(B : I) = Q_{[(S:A)]}^g$. Therefore, the integral homogeneous c-ideals of B are of the form $B_{[A]}$, where A is an integral c-ideal of S. Since B is a Krull order it follows that S has the ascending chain condition on integral c-ideals. Hence S is a Krull semigroup.

Further if A is an integral c-ideal of S, then by the previous $B_{[A]}$ is an integral c-ideal of B. Since $Cl_g^n(B) = \{1\}$ we obtain $B_{[A]} = Bn$ for some normalizing homogeneous element n of B. Say $n \in Q_a^g$, $a \in A$. Hence $B_{[A]} = B_{[Sa]}$, and thus $A = Sa$. Consequently, S is factorial. □

Proof of Theorem 1.2. Since S is factorial and $U(S) = \{e\}$, S is a free Abelian monoid. Let $\{x_i | i \in I\}$ be a free basis for S. Clearly $|I|$ is the torsion-free rank of S. As before, each $B_{[Sx_i]} \in G_g(B)$ and thus $B_{[Sx_i]} \cap R = R_{[Sx_i]} \in G_g(R)$. As $Cl_g^n(R) = \{1\}$ this implies $R_{[Sx_i]} = RX_i$ for some normalizing homogeneous element $X_i \in R$. Since $U(S) = \{e\}$ and because X_i is regular, it follows that $X_i \in R_{x_i}$ and $R_{x_i} = R_e X_i = X_i R_e$.

Let $s = x_1 \cdots x_n$ be an arbitrary nontrivial element of S. We prove by induction on n that $R_s = R_e X_1 \cdots X_n$. The case $n = 1$ has been shown above. Assume $n > 1$. Since $s \in Sx_1$, $R_s \subseteq R_{[Sx_1]} = RX_1$. Computing degrees it follows that $R_s = R_{x_2 \cdots x_n} X_1$. Hence the induction hypothesis yields $R_s = R_e X_1 \cdots X_n$. Consequently, R is the ring generated by $R_e \cup \{X_i | i \in I\}$. Since every X_i is a normalizing homogeneous element of R, $\sigma_i = X_i^{-1} - X_i$ (i.e. conjugation by X_i) defines an automorphism on R_e. It is then clear that $R \cong R_e * S = R_e[S, \sigma]$. This proves (2).

If moreover $Cl_g^c(R) = \{1\}$ then each X_i can be taken central. It follows that X_i commutes elementwise with R_e, and σ_i is the identity mapping. Hence $R = R_e[X_i | i \in I]$ a polynomial ring in commuting variables. □

Remark. In the proof of the theorem we actually did not need that the Krull order R has trivial graded normalizing class group. Essentially all that is needed is that all integral homogeneous c-ideals I of R with $P \cap R_e = 0$ are generated by a normalizing homogeneous element. Or equivalently, every homogeneous height one prime ideal P of R with $P \cap R_e = 0$ is in $P_g^n(R)$. Obviously, the latter is satisfied in case condition (2) holds.

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