PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 107, Number 2, October 1989

ON A THEOREM OF FEIT AND TITS

PETER B. KLEIDMAN AND MARTIN W. LIEBECK

(Communicated by Warren J. Wong)

ABSTRACT. Feit and Tits [3] lay the groundwork for determining the smallest degree of a projective representation of a finite extension of a finite simple group G. Provided G is not of Lie type in characteristic 2, they determine precisely when this degree is smaller than the degree of a projective representation of G itself. We complete this project by extending their results to the groups of Lie type in characteristic 2.

INTRODUCTION

In [3], Feit and Tits address the problem of finding the smallest degree of a nontrivial projective representation of a finite extension of a finite simple group G. Here, by a projective representation we mean a homomorphism to $\mathrm{PGL}_n(\mathbf{F})$ for some field $\mathbf{F}$ and some integer n, and by a finite extension of G we mean a finite group with a homomorphic image isomorphic to G. Let G be a nonabelian finite simple group and $\mathbf{F}$ an algebraically closed field, and define

$$R_{\mathbf{F}}(G) = \min\{n \mid G \text{ is contained in } \mathrm{PGL}_n(\mathbf{F})\}$$
$$M_{\mathbf{F}}(G) = \min\{n \mid \text{a finite extension of } G \text{ is contained in } \mathrm{PGL}_n(\mathbf{F})\}.$$

The interesting situation is the case in which $M_{\mathbf{F}}(G) < R_{\mathbf{F}}(G)$, and this is the subject under study in [3]. Indeed, Feit and Tits lay the groundwork for the classification of this situtation, and the purpose of our paper is to complete this classification. This appears in Theorem 3 below.

The symplectic groups $\mathrm{Sp}_{2n}(2)$ (with $n \geq 4$) provide examples in which $M_{\mathbf{F}}(G) < R_{\mathbf{F}}(G)$. Consider the group H_o described in [5, Theorem 5(b)], so that

$$H_o \cong (4 \circ 2^{1+2n})\cdot\mathrm{Sp}_{2n}(2).$$

The proof of [5, Theorem 5(b)] shows that H_o embeds in $\mathrm{GL}_{2^n}(\mathbf{F})$ for any algebraically closed field of odd characteristic or of characteristic 0, and hence $2^{2n}\cdot\mathrm{Sp}_{2n}(2)$ embeds in $\mathrm{PGL}_{2^n}(\mathbf{F})$. Consequently $M_{\mathbf{F}}(\mathrm{Sp}_{2n}(2)) \leq 2^n$. On the other hand, it follows from [10] that $R_{\mathbf{F}}(\mathrm{Sp}_{2n}(2)) > 2^n$, provided $n \geq 4$.

Received by the editors September 6, 1988 and, in revised form, October 3, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20C25.

Observe that if G is any simple subgroup of $\mathrm{Sp}_{2n}(2)$ and $\mathrm{char}(\mathbf{F}) \neq 2$, then $M_{\mathbf{F}}(G) \leq M_{\mathbf{F}}(\mathrm{Sp}_{2n}(2)) \leq 2^n$. With this is mind, let us define

$$n_G = \min\{n | G \text{ embeds in } \mathrm{Sp}_{2n}(2)\}.$$

We now see that $M_{\mathbf{F}}(G) \leq \min\{2^{n_G}, R_{\mathbf{F}}(G)\}$, provided $\mathrm{char}(\mathbf{F}) \neq 2$. One of the fundamental results in [3] is that equality holds here.

Theorem 1 [3]. *Assume that G is a nonabelian simple group, $\mathbf{F}$ is an algebraically closed field, and that $M_{\mathbf{F}}(G), R_{\mathbf{F}}(G)$ and n_G are as defined above.*

(i) *If* $\mathrm{char}(\mathbf{F}) = 2$, *then* $M_{\mathbf{F}}(G) = R_{\mathbf{F}}(G)$.
(ii) *If* $\mathrm{char}(\mathbf{F}) \neq 2$, *then* $M_{\mathbf{F}}(G) = \min\{2^{n_G}, R_{\mathbf{F}}(G)\}$.

Using Theorem 1, Feit and Tits [3, §4] then go on to show that any (then known) simple group G satisfying $M_{\mathbf{F}}(G) < R_{\mathbf{F}}(G)$ must be of Lie type in characteristic 2. Quoting the classification of finite simple groups, we can now state

Theorem 2 [3, §4]. *If $M_{\mathbf{F}}(G) < R_{\mathbf{F}}(G)$, then G is of Lie type in characteristic 2.*

Here we analyse the simple groups of Lie type in characteristic 2, and we determine precisely which simple groups G can in fact satisfy $M_{\mathbf{F}}(G) < R_{\mathbf{F}}(G)$. Our main result is

Theorem 3. *Let G be a nonabelian simple group and $\mathbf{F}$ an algebraically closed field. Then $M_{\mathbf{F}}(G) < R_{\mathbf{F}}(G)$ if and only if G and $\mathbf{F}$ appear in the following table:*

G	$\mathrm{char}(\mathbf{F})$	$R_{\mathbf{F}}(G) \geq$	$M_{\mathbf{F}}(G)$	*conditions*
$\mathrm{Sp}_{2n}(q)$, q *even*	$\neq 2$	$\frac{1}{2}q^{n-1}(q^{n-1}-1)(q-1)$	q^n	$n \geq 2$,
				$G \neq \mathrm{Sp}_4(2)', \mathrm{Sp}_6(2)$
$\Omega^{\pm}_{2n}(q)$, q *even*	$\neq 2$	$q^{n-1}(q^{n-2}-1)$	q^n	$n \geq 4$,
				$G \neq \Omega^{+}_{8}(2)$
$L_4(q)$, q *even*	$\neq 2$	q^3+q^2+q-1	q^3	$G \neq L_4(2)$
$G_2(q)$, $q = 2^{2e}$	$\neq 2, 3$	q^3+1	q^3	$G \neq G_2(4)$

Our strategy for showing $2^{n_G} \geq R_{\mathbf{F}}(G)$ (for those G not in the table) is to establish lower bounds for n_G using some modular representation theory to be found in [10, 11, 12], and to establish upper bounds for $R_{\mathbf{F}}(G)$ by finding suitably small representations appearing in [1, 2, 7, 13]. To prove that $2^{n_G} < R_{\mathbf{F}}(G)$ (for those G in the table) is for the most part straightforward, although we have to appeal to some results of [4, 6, 9, 14] in order to obtain lower bounds for $R_{\mathbf{F}}(G)$. In §2 we take care of various technical details which provide the bounds we require. The proof of Theorem 3 is then completed in §3.

2. Bounds for n_G and $R_{\mathbf{F}}(G)$

Throughout this section, $G = G(2^a) = G(q)$ denotes a simple group of Lie type over the field $\mathbf{F}_{2^a} = \mathbf{F}_q$, and $\mathbf{F}$ denotes an algebraically closed field. As a convenience, we sometimes write $L_d^+(q) = L_d(q)$, $L_d^-(q) = U_d(q)$, $E_6^+(q) = E_6(q)$ and $E_6^-(q) = {}^2E_6(q)$.

Proposition 4. *Write d_G for the smallest nontrivial 2-modular projective representation of G. Then d_G appears in the following table:*

G	d_G
$L_d^{\pm}(q)$	d
$\mathrm{Sp}_d(q)'$, $d \geq 4$	d *if* $G \neq \mathrm{Sp}_4(2)'$
	3 *if* $G = \mathrm{Sp}_4(2)'$
$\Omega_d^{\pm}(q)$, $d \geq 8$	d
$G_2(q)'$	6
${}^2B_2(q)$	4
${}^3D_4(q)$	8
$F_4(q)$, ${}^2F_4(q)'$	26
$E_6^{\pm}(q)$	27
$E_7(q)$	56
$E_8(q)$	248

Proof. This is well known. A proof can be found in [12, §2]. □

Assume that V is an $\mathbf{E}G$-module for some field $\mathbf{E}$. For a subfield $\mathbf{E}_o$ of $\mathbf{E}$, we say that V is *realized over* $\mathbf{E}_o$ if G acts on the $\mathbf{E}_o$-span of some $\mathbf{E}$-basis of V. In other words, the matrices corresponding to elements of G can be written with entries in $\mathbf{E}_o$.

Proposition 5. *Let V be an absolutely irreducible $\mathbf{F}_{2^b}G$-module which is realized over no proper subfield of $\mathbf{F}_{2^b}$, and let $d = d_G$ be as in Proposition 4.*

(i) *If G is untwisted or if G is of type ${}^2B_2$ or ${}^2F_4$, then $b|a$ and $\dim_{\mathbf{F}_{2^b}}(V) \geq d^{a/b}$.*

(ii) *If $G \cong U_d(2^a)$, $\Omega_d^-(2^a)$ or ${}^2E_6(2^a)$, then $b|2a$ and one of the following holds.*

(a) *$b \nmid a$ and $\dim_{\mathbf{F}_{2^b}}(V) \geq d^{2a/b}$.*

(b) *$b|a$ and $\dim_{\mathbf{F}_{2^b}}(V) \geq d^{a/b}$.*

Proof. Let k denote the algebraic closure of $\mathbf{F}_2$, and form the tensor product $\overline{V} = V \otimes k$. There is a corresponding homomorphism ρ from G to $\mathrm{GL}_m(k)$, where $m = \dim_k(\overline{V}) = \dim_{\mathbf{F}_{2^b}}(V)$. Write ν for the natural automorphism of $\mathrm{GL}_m(k)$ given by $t \mapsto t^2$ on matrix entries. Composing ρ with ν^i ($i = 0, 1, 2, \ldots$) gives rise to modules which we call $V^{(i)}$. Since the kG-module $\overline{V}$ is realized over $\mathbf{F}_{2^b}$, we have $\overline{V} \cong \overline{V}^{(b)}$. And since V is realized over no proper subfield of $\mathbf{F}_{2^b}$, whenever $\overline{V} \cong \overline{V}^{(i)}$, it follows that $b|i$.

Suppose first that G is untwisted or is of type ${}^2B_2$ or ${}^2F_4$. Then according to [15, p. 241], $\mathbf{F}_{2^a}$ is a splitting field for G. This means that $\overline{V}$ is realized over $\mathbf{F}_{2^a}$, and by the remark in the previous paragraph, $b|a$. It now follows from Theorems 2.1 and 2.3 in [11] that $m \geq d^{a/b}$, and so (i) is proved.

Now take the case where $G \cong U_d(2^a)$, $\Omega_d^-(2^a)$ or ${}^2E_6(2^a)$. Here [15, p. 241] implies that $\mathbf{F}_{2^{2a}}$ is a splitting field, whence $b|2a$. Obviously $a/(a,b) = a/b$ if $b|a$, and $a/(a,b) = 2a/b$ if $b \nmid a$, and so (ii) follows directly from Theorem 2.2 of [11]. □

We are now in a position to obtain lower bounds for n_G. In the proof of Proposition 6 below, it is convenient to quote the theorem of Zsigmondy [16] which says the following. Let x, y be integers with $x \geq 2$ and $y \geq 3$. Then provided $(x, y) \neq (2, 6)$, there exists a prime divisor p of $x^y - 1$ such that p does not divide $x^z - 1$ for $1 \leq z \leq y - 1$. Such a prime is called a *primitive prime divisor* of $x^y - 1$.

Proposition 6. *Let $d = d_G$ be as in Proposition 4. A lower bound for n_G appears in the following table*:

G	$n_G \geq$
$L_2(2^a)$	a
$L_d^{\pm}(2^a)$, $d \geq 3$	da *if* $d \neq 4$
	$3a$ *if* $d = 4$
$\mathrm{Sp}_d(2^a)'$, $d \geq 4$	$da/2$
$\Omega_d^{\pm}(2^a)$, $d \geq 8$	$da/2$
$G_2(2^a)'$	$3a$
${}^2B_2(2^a)$	$2a$
${}^3D_4(2^a)$	$6a$
$F_4(2^a)$, ${}^2F_4(2^a)'$	$13a$
$E_6^{\pm}(2^a)$	$27a/2$
$E_7(2^a)$	$28a$
$E_8(2^a)$	$124a$

Proof. Write $G \leq \mathrm{Sp}_{2n_G}(2)$, so that G acts on the natural $2n_G$-dimensional module over $\mathbf{F}_2$. Let V be a G-invariant section of this module upon which G

acts faithfully and irreducibly. Write $m = \dim_{\mathbf{F}_2}(V)$, so that $G \leq \mathrm{GL}(V, \mathbf{F}_2) \cong \mathrm{GL}_m(2)$. Now there is a divisor b of m such that as an $\mathbf{F}_{2^b}G$-module,

$$V \otimes \mathbf{F}_{2^b} \cong \bigoplus_{\sigma \in \mathrm{Gal}(\mathbf{F}_{2^b}:\mathbf{F}_2)} W^\sigma,$$

where W is an absolutely irreducible $\mathbf{F}_{2^b}G$-module with $\dim_{\mathbf{F}_{2^b}}(W) = m/b$ (see [8, Theorem 9.21] for example). Moreover, if $\sigma, \tau \in \mathrm{Gal}(\mathbf{F}_{2^b}:\mathbf{F}_2)$ with $\sigma \neq \tau$, then $W^\sigma \ncong W^\tau$ as $\mathbf{F}_{2^b}$-modules, and it follows that W is realized over no proper subfield of $\mathbf{F}_{2^b}$. We are now in a position to exploit Proposition 5.

First assume that G is of type $L_2, \mathrm{Sp}_d, \Omega_d^+, G_2, {}^2B_2, {}^2F_4, F_4, E_6, E_7$ or E_8. Then it follows from Proposition 5 that $b|a$ and $m/b \geq d^{a/b}$. Hence $2n_G \geq m \geq bd^{a/b} \geq da$, as required.

Assume here that $G \cong L_d(2^a)$ with $d \geq 3$. As in the previous paragraph, we know that $b|a$. Consider first the case in which $a = b$. Here $m/a = m/b \geq d_G = d$. Now if V is not a self-dual G-module, it follows that $2n_G \geq 2m \geq 2da$, as desired. We may assume therefore that V is self-dual, and hence so is $V \otimes \mathbf{F}_{2^a}$. Consequently $W^* \cong W^\sigma$ for some $\sigma \in \mathrm{Gal}(\mathbf{F}_{2^a}:\mathbf{F}_2)$, and hence W is either self-dual or unitary according as $|\sigma| = 1$ or 2. Since the irreducible d-dimensional modules for $\mathrm{SL}_d(2^a)$ are neither self-dual or unitary, we have $m/a > d$. Therefore by [11, Theorem 1.1], $m/a \geq \frac{1}{2}d(d-1)$. So if $d \geq 5$, then $m \geq 2da$, and the desired result follows. If $d = 4$, we have $m \geq 6a$, whence $n_G \geq 3a$. If $d = 3$, then it follows from [12, Theorem 2.2] that there are no irreducible $\mathbf{F}_{2^b}L_3(2^a)$-modules of dimension 4 or 5, and so once again $m \geq 6a$, which proves $n_G \geq 3a$. We now consider the case where $b < a$. Here $m/b \geq d^{a/b}$, and so with only one exception, $m \geq bd^{a/b} \geq 2da$, as desired. The one exception is the case $d = 3$ and $a = 2b$. To treat this exceptional case, observe that the proof of Proposition 5(i) (see the proof of [11, Theorem 2.1]) actually shows that $W \otimes \mathbf{F}_{2^a} \cong U \otimes U^\tau$, where U is an absolutely irreducible $\mathbf{F}_{2^a}\mathrm{SL}_3(2^a)$-module and τ generates $\mathrm{Gal}(\mathbf{F}_{2^a}:\mathbf{F}_{2^b})$. Write $d_1 = \dim_{\mathbf{F}_{2^a}}(U)$, so that $m/b = d_1^2$. If $d_1 \geq 4$, then $2n_G \geq m \geq 16b = 8a > 6a = 2da$, which is the required result. And if $d_1 = 3$, then U is neither self-dual nor unitary, and hence the same holds of W; hence as we argued before, V is not self-dual. Therefore $n_G \geq m \geq 9b = 9a/2 > 3a$, as claimed.

Next suppose that $G \cong \Omega_d^-(2^a)$, with d even and $d \geq 8$, so that $b|2a$ by Proposition 5. If b does not divide a, then $m/b \geq d^{2a/b}$, and so $m \geq 2ad$, as desired. And if $b|a$, then as before we obtain $m/b \geq d^{a/b}$, and hence $m \geq da$.

Assume here that $G \cong U_d(2^a)$ with $d \geq 3$. If $d = 3$, then $a \geq 2$ (since $U_3(2)$ is solvable), and $2^{3a} + 1 \mid |G|$. Thus $|G|$ is divisible by a primitive prime divisor of $2^{6a} - 1$, and so $m \geq 6a$, as claimed. We assume for the rest of this paragraph that $d \geq 4$. By Proposition 5, $b|2a$. Moreover, if b does not divide a, we may argue as in the previous paragraph. Assume therefore that $b|a$. If $b < a$, then $m/b \geq d^{a/b}$, and so $m \geq bd^{a/b} \geq 2da$ (since $d \geq 4$). It remains to consider the case $a = b$. Then by [11, Theorem 2.2], we have

$$m/b = m/a \geq \begin{cases} d(d-1) & \text{if } d \geq 7 \\ 20 & \text{if } 5 \leq d \leq 6 \\ 6 & \text{if } d = 4. \end{cases}$$

So provided $d \geq 5$, we have $m/a > 2d$, whence $n_G > da$. When $d = 4$, we deduce $m/a \geq 6$, whence $n_G \geq 3a$.

If $G \cong {}^2E_6(2^a)$, then as in the previous two paragraphs, we see that $m \geq 27a$, and so $n_G \geq 27a/2$.

Finally, assume that $G \cong {}^3D_4(2^a)$. Then $2^{4a} - 2^{2a} + 1 \mid |G|$, and so G is divisible by a primitive prime divisor of $2^{12a} - 1$. This forces $m \geq 12a$, and the result follows. □

At this stage we provide upper bounds for $R_{\mathbf{F}}(G)$.

Proposition 7. *Apart from* $\mathrm{Sp}_d(q)$ *and* $\Omega_d^{\pm}(q)$, *upper bounds for* $R_{\mathbf{F}}(G)$ *are given in the following table*:

G	$R_{\mathbf{F}}(G) \leq$
$L_d^{\pm}(q)$	$(q^d - 1)/(q \mp 1)$
$G_2(q)'$	$q^3 + \varepsilon,\ \varepsilon = \pm 1,\ q \equiv \varepsilon \pmod 3$
${}^2B_2(q)$	$\sqrt{q/2}(q-1)$
${}^3D_4(q)$	$q(q^4 - q^2 + 1)$
${}^2F_4(q)'$	$\sqrt{q/2}(q^2-1)(q^3+1)$
$F_4(q)$	$\frac{1}{2}q(q+1)^2(q^2-q+1)^2(q^4+1)$
$E_6^{\pm}(q)$	$q(q^4 \mp 1)(q^6 \pm q^3 + 1)$
$E_7(q)$	$q(q^6+q^5+q^4+q^3+q^2+q+1)\times$
	$(q^4-q^2+1)(q^6-q^5+q^4-q^3+q^2-q+1)$
$E_8(q)$	$q(q^2+1)^2(q^4+1)(q^4-q^2+1)\times$
	$(q^8-q^6+q^4-q^2+1)(q^8-q^4+1)$

Proof. It suffices to exhibit a faithful ordinary character of a covering group of G which has degree less than or equal to the number given in the right-hand column. For $L_d(q)$ this is easy, as it has a permutation representation of degree $(q^d - 1)/(q-1)$, and hence has a faithful ordinary character of degree 1 less. The result for $U_d(2^a)$ appears in [13], and for the exceptional groups apart from $G_2(q)$ one can appeal to [1, §13.9]. For $G_2(q)$ we appeal to [2] or [14, pp. 293-4]. □

3. The proof of Theorem 3

Assume initially that G and $\mathbf{F}$ are as given in the table appearing in the statement of Theorem 3. We must show that $M_{\mathbf{F}}(G) < R_{\mathbf{F}}(G)$, and to do so it suffices to verify the values of $M_{\mathbf{F}}(G)$ and the bounds for $R_{\mathbf{F}}(G)$ displayed in the table. The bounds for $R_{\mathbf{F}}(G)$ when G is $\mathrm{Sp}_{2n}(q)$ or $\Omega_{2n}^{\varepsilon}(q)$ are taken from [10]. When $G = L_4(q)$ with $q \geq 4$, it follows from [4] that the smallest ordinary faithful character degree of G is $q(q^2+q+1)$. It can be seen using the decomposition matrices for $\mathrm{GL}_4(q)$ in [9] that $R_{\mathbf{F}}(G) \geq q(q^2+q+1)-1$. And when G is $G_2(q)$, $q = 4^k > 4$, it is shown in [6, 14] that $R_{\mathbf{F}}(G) = q^3+1$ (n.b., $\mathrm{char}(\mathbf{F}) \geq 5$). Now when $G = \mathrm{Sp}_{2n}(q)$ with $q = 2^a$, it is obvious that $G < \mathrm{Sp}_{2na}(2)$, and so $n_G \leq na$. Consequently $n_G = na$ by Proposition 6, and as $2^{n_G} = q^n < R_{\mathbf{F}}(G)$, we deduce from Theorem 1 that $M_{\mathbf{F}}(G) = q^n$. In the exact same fashion we may establish $M_{\mathbf{F}}(\Omega_{2n}^{\pm}(q)) = q^n$ for $n \geq 4$. Also $L_4(2^a) \cong \Omega_6^+(2^a) < \mathrm{Sp}_6(2^a) < \mathrm{Sp}_{6a}(2)$ and $G_2(2^a) < \mathrm{Sp}_{6a}(2)$, and hence by Proposition 6 we have $n_G = 3a$ for these groups. Thus $M_{\mathbf{F}}(L_4(q)) = M_{\mathbf{F}}(G_2(q)) = q^3$.

We now prove the converse. Assume that G is a nonabelian simple group, $\mathbf{F}$ an algebraically closed field, and assume that $M_{\mathbf{F}}(G) < R_{\mathbf{F}}(G)$. Our goal is to show that G and $\mathbf{F}$ appear in the table given in Theorem 3. Now according to Theorem 1, $\mathrm{char}(\mathbf{F}) \neq 2$ and

$$(*) \qquad 2^{n_G} < R_{\mathbf{F}}(G).$$

Also, Theorem 2 asserts that G is of Lie type in characteristic 2. Suppose first that $G \cong \mathrm{Sp}_{2n}(q)'$ with $n \geq 2$. As $R_{\mathbf{F}}(\mathrm{Sp}_4(2)') \leq 3$ and $R_{\mathbf{F}}(\mathrm{Sp}_6(2)) = 7$, it follows that G is not one of these, and so G does indeed appear in the table. And if $G \cong \Omega_{2n}^{\pm}(q)$ with $n \geq 4$, then G is not $\Omega_8^+(2)$, for $R_{\mathbf{F}}(\Omega_8^+(2)) = 8$, while $n_{\Omega_8^+(2)} = 4$. So once again, G appears in the table. Thus we can assume hereafter that G is neither a symplectic group nor an orthogonal group. It now follows from $(*)$ and Propositions 6 and 7 that G is $L_2(q)$, $L_4(q)$ or $G_2(q)$. Now if $G \cong L_2(q)$, then $2^{n_G} = q$; however it is well known that $R_{\mathbf{F}}(G) \leq q-1$. Thus this case cannot arise. And if $G \cong L_4(2)$, then $2^{n_G} = 8$, while $R_{\mathbf{F}}(G) \leq 7$. So this case cannot arise either, and so G does indeed appear in the table if $G \cong L_4(q)$. It remains to consider $G \cong G_2(q)$, so that $2^{n_G} = q^3$ by Proposition 6. It follows from [2] that if $\log_2(q)$ is odd, then $R_{\mathbf{F}}(G) \leq q^3-1$. Thus it must be the case that $\log_2(q)$ is even. The group $G_2(4)$ has an exceptional multiplier and the covering group $2.G_2(4)$ has a faithful representation of degree 14. Thus we must have $G \cong G_2(2^{2e})$ with $e \geq 2$. Now according to [7], $R_{\mathbf{F}}(G) \leq q^3$ if $\mathrm{char}(\mathbf{F}) = 3$, and we conclude that $\mathrm{char}(\mathbf{F}) \neq 2, 3$. Thus the proof is finally complete.

Acknowledgment

The authors thank Professor G. Hiss for communicating to us his results on the modular representations of $G_2(q)$.

References

1. R. W. Carter, *Finite groups of Lie type, Conjugacy Classes and Complex Characters*, J. Wiley and Sons, 1985.
2. H. Enomoto and H. Yamada, *The characters of* $G_2(2^n)$, Japan J. Math. (N.S.) **12** (1986), 325–377.
3. W. Feit and J. Tits, *Projective representations of minimum degree of group extensions*, Canad. J. Math. **30** (1978), 1092–1102.
4. J. A. Green, *The characters of the finite general linear groups*, Trans. Amer. Math. Soc. **80** (1955), 402–447.
5. R. L. Griess, Jr., *Automorphisms of extra special groups and nonvanishing degree* 2 *cohomology*, Pacific J. Math. **48** (1973), 403–422.
6. G. Hiss, *On the decomposition numbers of* $G_2(q)$, J. Algebra **120** (1989), 339–360.
7. G. Hiss and J. Shamash, *The* 3*-modular representations of* $G_2(2^n)$ (to appear).
8. I. M. Isaacs, *Character theory of finite groups*, Academic Press, New York, 1976.
9. G. D. James, *The decomposition matrices for* $\mathrm{GL}_n(q)$, $n \leq 10$, Imperial College, London, (preprint).
10. V. Landazuri and G. M. Seitz, *On the minimal degrees of projective representations of the finite Chevalley groups*, J. Algebra **32** (1974), 418–443.
11. M. W. Liebeck, *On the orders of maximal subgroups of the finite classical groups*, Proc. London Math. Soc. (3) **50** (1985), 426–446.
12. M. W. Liebeck, *The affine permutation groups of rank three*, Proc. London Math. Soc. (3) **54** (1987), 447–516.
13. G. M. Seitz, *Some representations of classical groups*, J. London Math. Soc. **10** (1975), 115–120.
14. J. Shamash, *Blocks and Brauer trees for groups of type* $G_2(q)$, Proc. Sympos. Pure Math. **47** Part 2 (1987), 283–295.
15. R. Steinberg, *Lectures on Chevalley groups*, Yale University Mathematics Department, 1968.
16. K. Zsigmondy, *Zur theorie der Potenzreste*, Monatsh. Mat. und Phys. **3** (1892), 265–284.

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CALIFORNIA 91125

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE OF SCIENCE AND TECHNOLOGY, LONDON, SW7 2AZ ENGLAND