PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 107, Number 2, October 1989

PERIODIC POINT FREE HOMEOMORPHISM OF T^2

MICHAEL HANDEL

(Communicated by Kenneth R. Meyer)

ABSTRACT. Suppose that $f: T^2 \to T^2$ is an orientation preserving homeomorphism of the torus that is homotopic to the identity and that has no periodic points. We show that there is a direction θ and a number ρ such that every orbit of f has rotation number ρ in the direction θ.

1. INTRODUCTION

The rotation number of an orientation preserving circle homeomorphism is an essential tool for analyzing its dynamics. For orientation preserving homeomorphisms $f: T^n \to T^n$ $(n > 1)$ of higher dimensional tori that are homotopic to the identity, there is a straightforward generalization of the rotation number that we may consider ([He]). Namely, for each $x \in T^n$, choose lifts $\tilde{x} \in \mathbf{R}^n$ and $\tilde{f}: \mathbf{R}^n \to \mathbf{R}^n$ to the universal cover and define the translation vector $\tau(\tilde{f}, \tilde{x})$ to be the limit, if it exists, of the average displacement vector $(\tilde{f}^n(\tilde{x}) - \tilde{x})/n$. If $\tau(\tilde{f}, \tilde{x})$ is well-defined, then the rotation vector $\rho(f, x) \in T^n$ of the f-orbit of x is the projected image of $\tau(\tilde{f}, \tilde{x})$.

The rotation number can also be computed as an average $\rho(f, x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \varphi(f^i(x))$ where $\varphi: T^n \to T^n$ is the displacement function $\varphi(y) = f(y) - y$. The ergodic theorem therefore implies that $\rho(f, x)$ is defined on a set of full μ-measure for any f-invariant measure μ. Moreover, if f is uniquely ergodic then $\rho(f, x)$ is defined for all x and in fact is independent of x.

Herman [He] has shown that there are orientation preserving minimal diffeomorphisms $f: T^n \to T^n$ $(n \geq 3)$ for which not every $\rho(f, x)$ is well defined. Thus any hope of finding a sufficient topological condition to guarantee the existence of a rotation vector at each point in T^n is restricted to the case $n = 2$.

In this paper we show that if $f: T^2 \to T^2$ is orientation preserving, homotopic to the identity and periodic point free (see Remark 1.3 for a weakening of this condition) then "half" of a rotation vector is defined. More precisely, for any vector $\vec{v} \in \mathbf{R}^2$, define $\tau(\tilde{f}, \tilde{x}, \vec{v})$ to be the limit, if it exists, of the projection of the average displacement vector $(\tilde{f}^n(\tilde{x}) - \tilde{x})/n$ onto $\vec{v}$. If $\tau(\tilde{f}, \tilde{x})$ is well-defined and $\vec{v}$ is a unit vector then $\tau(\tilde{f}, \tilde{x}, \vec{v})$ equals the scalar product

Received by the editors July 3, 1988 and, in revised form, October 5, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54H20, 57M99.

$\tau(\tilde{f}, \tilde{x}) \cdot \vec{v}$. Conversely if $\tau(\tilde{f}, \tilde{x}, \vec{v}_1)$ and $\tau(\tilde{f}, \tilde{x}, \vec{v}_2)$ are defined for linearly independent $\tilde{v}_1$ and $\tilde{v}_2$, then $\tau(\tilde{f}, \tilde{x})$ is well-defined and uniquely determined by $\tau(\tilde{f}, \tilde{x}, \vec{v}_1)$ and $\tau(\tilde{f}, \tilde{x}, \vec{v}_2)$.

Theorem 1.1. *If $f: T^2 \to T^2$ is an orientation preserving periodic point free homeomorphism that is homotopic to the identity, then there exists a vector $\vec{v} \in \mathbf{R}^n$ such that $\tau(\tilde{f}, \tilde{x}, \vec{v})$ is defined for all lifts $\tilde{f}: \mathbf{R}^n \to \mathbf{R}^n$ and $\tilde{x} \in \mathbf{R}^n$; moreover $\tau(\tilde{f}, \tilde{x}, \vec{v})$ is independent of $\tilde{x}$.*

Remark 1.2. Skew products $f(\theta_1, \theta_2) = (\theta_1 + \alpha, g(\theta_1, \theta_2))$ where $\alpha \in S^1$ and $g: T^2 \to S^1$ are good examples to keep in mind. In this case $\vec{v}$ is tangent to the line that projects onto the θ_1-circle in T^2 and $\overline{\tau}(\tilde{f}, \tilde{x}, \vec{v}) \bmod 1 = \alpha$. Herman [He] has shown that $\rho(f, x)$ always exists for these f. If α is irrational then $\rho(f, x)$ is independent of x but if α is rational then $\rho(f, x)$ may vary.

Remark 1.3. Theorem 1.1 is a special case of a more general result about zero entropy surface diffeomorphisms. Using techniques similar to those in [Ha, Section 9], one can show for example that the conclusions of Theorem 1.1 remain valid if one only assumes that f has zero topological entropy and a finite but non-empty fixed point set. We have concentrated on the periodic point free case in this paper because the techniques are very different from those of the zero entropy case and because, as the following example (shown to the author by A. B. Katok) demonstrates, rotation vectors do not always exist if f has fixed (or periodic) points. It is an open question as to whether or not $\rho(f, x)$ is defined for all x when f is periodic point free.

Example 1.4. Let Y be a unit vector field on T^2 that generates an irrational flow in the direction $\vec{w}$. Choose a point $P \in T^2$ and a function $u: T^2 \to [0, 1]$ such that $u^{-1}(0) = P$ and such that the flow φ_t generated by $u \cdot Y$ is not uniquely ergodic. In particular, there is an f-invariant ergodic measure μ such that $\int u \, d\mu > 0$.

Let $S = \{x \in T^2 : \Phi(x) = \lim_{n\to\infty} \frac{1}{T} \int_0^T u(\varphi_t(x))\, dt$ is well defined$\}$. The ergodic theorem implies that $A = \{x \in S : \Phi(x) > (\int u\, d\mu)/2\}$ is non-empty. The set $B = \{x \in S : \Phi(x) = 0\}$ is also non-empty since it contains the half-infinite flow line whose forward end is P. As Φ is constant on flow lines, A and B are both dense in T^2. It follows that $S \neq T^2$. For $y \in T^2 - S$ and lifts $\tilde{y}$ of y and $\tilde{f}$ of $f = \varphi_1$, $\tau(\tilde{f}, \tilde{y}, \vec{w})$ is not defined.

2. Proof of Theorem 1.1

Let $T_1(x, y) = (x + 1, y)$ and $T_2(x, y) = (x, y + 1)$ be the covering translations of $\mathbf{R}^2$; Let $p_1, p_2: \mathbf{R}^2 \to \mathbf{R}^2$ be the projections onto the first and second coordinate factors respectively.

For any lift $\tilde{g}: \mathbf{R}^2 \to \mathbf{R}^2$ of an orientation preserving homeomorphism $g: T^2 \to T^2$, we define functions $\alpha(\tilde{g}, \tilde{x}, \{n_k\})$ and $\beta(\tilde{g}, \tilde{x}, \{n_k\})$ where $\tilde{x} \in \mathbf{R}^2$

and where $\{n_k\}$ is a sequence of positive integers such that *both* limits

$$\alpha(\tilde{g}, \tilde{x}, \{n_k\}) = \lim_{k\to\infty} (p_1 \tilde{g}^{n_k}(\tilde{x}) - p_1(\tilde{x}))/n_k$$

and

$$\beta(\tilde{g}, \tilde{x}, \{n_k\}) = \lim_{k\to\infty} (p_2 \tilde{g}^{n_k}(\tilde{x}) - p_2(\tilde{x}))/n_k$$

exist.

If either $\alpha(\tilde{g}, \tilde{x}, \{n_k\}) \neq 0$ or $\beta(\tilde{g}, \tilde{x}, \{n_k\}) \neq 0$, then we may define $\theta(\tilde{g}, \tilde{x}, \{n_k\})$ to be the polar coordinate angle of the vector $\langle \alpha(\tilde{g}, \tilde{x}, \{n_k\}), \beta(\tilde{g}, \tilde{x}, \{n_k\}) \rangle$. Define $\Theta(\tilde{g})$ to be the union of all $\theta(\tilde{g}, \tilde{x}, \{n_k\})$ as $\tilde{x}$ and $\{n_k\}$ vary. If $\tilde{x}$ projects to a periodic point of g, then $\theta(\tilde{g}, \tilde{x}, \{n_k\})$ is independent of $\{n_k\}$ and we will sometimes write $\theta(\tilde{g}, \tilde{x})$ in this circumstance.

The following proposition was inspired by an argument of Franks [Fr].

Proposition 2.1. *If $\tilde{g}: \mathbf{R}^2 \to \mathbf{R}^2$ is fixed point free, then $\Theta(\tilde{g})$ is contained in a semi-circle.*

Proof. Assume that $\tilde{g}$ is fixed point free. Fix $\theta = \theta(\tilde{g}, \tilde{x}, \{n_k\}) \in \Theta(\tilde{g})$ and $0 < \varepsilon < \min_{\tilde{x}\in\mathbf{R}^2} |\tilde{g}(\tilde{x}) - \tilde{x}|$. After passing to a subsequence of $\{n_k\}$ we may assume that $|g^{n_k}(x) - g^{n_l}(x)| < \varepsilon$ for all k and l where $x \in T^2$ is the projected image of $\tilde{x}$. Choose $k \gg l$ and let $g_1: T^2 \to T^2$ be a homeomorphism that is ε-close to the identity, that carries $g^{n_k}(x)$ to $g^{n_l}(x)$ and that fixes $g^i(x)$ for $n_l < i < n_k$. Then $h = g_1 \circ g$ is ε-close to g in the C^0-topology and there is a lift $\tilde{h}$ that is ε-close to $\tilde{g}$ in the C^0-topology. In particular $\tilde{h}$ is fixed point free. The point $y = g^{n_l}(x)$ is h-periodic and if k is sufficiently large relative to l, then $\theta(\tilde{h}, \tilde{y})$ is ε-close to θ.

This procedure can be applied simultaneously to any three points θ_1, θ_2 and θ_3 in $\Theta(\tilde{g})$. (If $\theta_1 = \theta(\tilde{g}, \tilde{x}_1, \{n_k\})$ and $\theta_2 = \theta(\tilde{g}, \tilde{x}_2, \{m_l\})$ where $\tilde{x}_1$ and $\tilde{x}_2$ project onto the same point $x \in T^2$, then y_1 and y_2 are obtained by closing distinct g-orbits of points that are close to x.) Thus for all $\varepsilon > 0$ and all $\theta_1, \theta_2, \theta_3 \in \Theta(\tilde{g})$, there exists a homeomorphism $h: T^2 \to T^2$ that is ε-close to g in the C^0-topology and that has periodic orbits y_1, y_2 and y_3 satisfying $|\theta(\tilde{h}, \tilde{y}_i) - \theta_i| < \varepsilon$. Radially conjugate $\tilde{h}: \mathbf{R}^2 \to \mathbf{R}^2$ to a homeomorphism $H: \operatorname{int} D^2 \to \operatorname{int} D^2$, and label by z_i the points in $\operatorname{int} D^2$ corresponding to $\tilde{y}_i$ under the conjugating map. Extend H over ∂D^2 by the identity. Then the hypotheses of Lemma 2.2 below are satisfied with ω_i and α_i being antipodal points and with $|\omega_i - \theta_i| < \varepsilon$. We conclude that the θ_i's and $(\theta_i + \pi)$'s do not alternate on S^1 and hence that the θ_i''s lie in a semi-circle. □

Lemma 2.2. *Suppose that $H: D^2 \to D^2$ is an orientation preserving homeomorphism of the closed disk and that $y_1, y_2, y_3 \in \operatorname{int} D^2$ have α-limit sets $\alpha_1, \alpha_2, \alpha_3$ and ω-limits sets $\omega_1, \omega_2, \omega_3$ that are single points on ∂D^2. If the α_i's and ω_i's alternate around ∂D^2 then $H|\operatorname{int} D^2$ has a fixed point.*

The proof of Lemma 2.2 is contained in [Ha, Section 5 (Proposition 5.4)]. Recall that the Brouwer translation theorem implies that if $h\colon \mathbf{R}^2 \to \mathbf{R}^2$ is a fixed point free orientation preserving homeomorphism then each h-orbit is without accumulation points. One may intuitively think of Lemma 2.2 as stating that if $h\colon \mathbf{R}^2 \to \mathbf{R}^2$ has orbits that move in directions that do not all lie in some half-plane, then some orbit of h has bounded forward orbit.

Proof of Theorem 1.1. Choose a lift $\tilde{f}\colon \mathbf{R}^2 \to \mathbf{R}^2$ of $f\colon T^2 \to T^2$. We will show that there are constants A, B and C such that

$$\begin{aligned}\langle A, B\rangle \cdot \langle \alpha(\tilde{f}, \tilde{x}, \{n_k\}), \beta(\tilde{f}, \tilde{x}, \{n_k\})\rangle \\ = A\alpha(\tilde{f}, \tilde{x}, \{n_k\}) + B\beta(\tilde{f}, \tilde{x}, \{n_k\}) = C\end{aligned}$$

for all $\tilde{x}$ and $\{n_k\}$ (for which $\alpha(\tilde{f}, \tilde{x}, \{n_k\})$ and $\beta(\tilde{f}, \tilde{x}, \{n_k\})$ are defined). This implies that $\tau(\tilde{f}, \tilde{x}, \vec{v})$ is well-defined and independent of $\tilde{x}$ where $\vec{v}$ is the vector $\langle A, B\rangle$.

The functions α and β behave well when $\tilde{f}$ is replaced by any lift of any positive iterate of f. Namely,

$$\alpha(T_1^{-p} T_2^{-r} \tilde{f}^q, \tilde{x}, \{[n_k/q]\}) = q\alpha(\tilde{f}, \tilde{x}, \{n_k\}) - p$$

and

$$\beta(T_1^{-p} T_2^{-r} \tilde{f}^q, \tilde{x}, \{[n_k/q]\}) = q\beta(\tilde{f}, \tilde{x}, \{n_k\}) - r$$

where [ ] is the greatest integer function and $q > 0$.

If $\beta(\tilde{f}, \tilde{x}, \{n_k\})$ is independent of $\tilde{x}$ and $\{n_k\}$, choose $A = 0, B = 1$ and $C = \beta(\tilde{f}, \tilde{x}, \{n_k\})$. Otherwise we may assume that β takes on at least two values $\beta_1 = \beta(\tilde{f}, \tilde{x}_1, \{u_i\}) < \beta(\tilde{f}, \tilde{x}_2, \{w_j\}) = \beta_2$. Replacing $\tilde{f}$ by $T_2^{-r}\tilde{f}^q$ where $\beta_1 < r/q < \beta_2$ if necessary, we may assume that $\beta_1 < 0 < \beta_2$. Define $\alpha_1 = \alpha(\tilde{f}, \tilde{x}_1, \{u_i\})$ and $\alpha_2 = \alpha(\tilde{f}, \tilde{x}_2, \{w_j\})$.

As $p/q \to \gamma = (\beta_2\alpha_1 - \beta_1\alpha_2)/(\beta_2 - \beta_1)$, $\theta(T_1^{-p}\tilde{f}^q, \tilde{x}_1, \{[u_i/q]\})$ and $\theta(T_1^{-p}\tilde{f}^q, \tilde{x}_2, \{[w_j/q]\})$ approach distinct (because $\beta_2 > 0 > \beta_1$) angles φ_1 and φ_2 whose tangents $\beta_1/(\alpha_1 - \gamma)$ and $\beta_2/(\alpha_2 - \gamma)$ are equal. Thus φ_1 and φ_2 are antipodal. The points $\theta(T_1^{-p}\tilde{f}^q, \tilde{x}_1, \{[u_i/q]\})$ and $\theta(T_1^{-p}\tilde{f}^q, \tilde{x}_2, \{[w_j/q]\})$ lie in the interior of the semi-circle that has endpoints $\{\varphi_1, \varphi_2\}$ and that contains 0 for $p/q > \gamma$ and in the interior of the semi-circle that has endpoints $\{\varphi_1, \varphi_2\}$ and that contains π for $p/q < \gamma$.

Choose any $\tilde{x}$ and $\{n_k\}$ such that $\alpha(\tilde{f}, \tilde{x}, \{n_k\})$ and $\beta(\tilde{f}, \tilde{x}, \{n_k\})$ are defined. If $\alpha(\tilde{f}, \tilde{x}, \{n_k\}) \neq \gamma$ or if $\beta(\tilde{f}, \tilde{x}, \{n_k\}) \neq 0$, then $\theta(T_1^{-p}\tilde{f}^q, \tilde{x}, \{[n_i/q]\})$ is defined for all p/q close to γ and is a uniformly continuous function of p/q. Applying Proposition 2.1 to p/q close to and on both sides of γ, we conclude that $\varphi = \lim_{p/q \to \gamma} \theta(T_1^{-p}\tilde{f}^q, \tilde{x}, \{[n_i/q]\})$ is either φ_1 or φ_2. In particular,

$$\tan\varphi = (\beta(\tilde{f}, \tilde{x}, \{n_k\})/(\alpha(\tilde{f}, \tilde{x}, \{n_k\}) - \gamma)$$

is independent of $\tilde{x}$ and $\{n_k\}$. Choose $A = \tan\varphi, B = -1$ and $C = \gamma\tan\varphi$. Note that $A\alpha(\tilde{f}, \tilde{x}, \{n_k\}) + B\beta(\tilde{f}, \tilde{x}, \{n_k\}) = C$ even when $\alpha(\tilde{f}, \tilde{x}, \{n_k\}) = \gamma$ and $\beta(\tilde{f}, \tilde{x}, \{n_k\}) = 0$.

REFERENCES

[Fr] Franks, John, *Recurrence and fixed points of surface homeomorphisms*, to appear in Ergodic Theory and Dynamical Systems.

[Ha] Handel, M., *Zero entropy surface diffeomorphisms*, preprint .

[He] Herman, M., *Une methode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractere local d'un theoreme d'Arnold et de Moser sur le tore de dimension* 2, Comment. Math. Helv. **58** (1983), no. 3, 453–502.

DEPARTMENT OF MATHEMATICS, LEHMAN COLLEGE, BRONX, NEW YORK 10468