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GRAPHS WITH PARALLEL MEAN CURVATURE

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ABSTRACT. We prove that if the graph $\Gamma_f = \{(x, f(x)) : x \in M\}$ of a map $f: (M, g) \to (N, h)$ between Riemannian manifolds is a submanifold of $(M \times N, g \times h)$ with parallel mean curvature H, then on a compact domain $D \subset M$, $\|H\|$ is bounded from above by $\frac{1}{m}\frac{A(\partial D)}{V(D)}$. In particular, Γ_f is minimal provided M is compact, or noncompact with zero Cheeger constant. Moreover, if M is the m-hyperbolic space—thus with nonzero Cheeger constant—then there exist real-valued functions the graphs of which are nonminimal submanifolds of $M \times \mathbf{R}$ with parallel mean curvature.

1. INTRODUCTION

Let $f: M \to N$ be a smooth map, where M, N are Riemannian manifolds of dimensions m, n and Riemannian metrics g, h, respectively. The graph of f, $\Gamma_f = \{(x, f(x)) : x \in M\}$, is a submanifold of $M \times N$ of dimension m. We take on $M \times N$, the product metric, and on Γ_f, the induced one. The purpose of this paper is to prove that if Γ_f is a submanifold with parallel mean curvature, then actually Γ_f is minimal provided M is compact, or noncompact with zero Cheeger constant (see Theorems 1 and 2 in §2). Furthermore, the behavior of the mean curvature of a graph is studied in some special cases in §3.

This problem of estimating the mean curvature of a graph was first introduced in 1955 by E. Heinz [7] for the case of a map $f: \mathbf{R}^2 \to \mathbf{R}$. He proved that if $z = z(x, y)$ is a surface of $\mathbf{R}^3$ defined for $x^2 + y^2 < R^2$ with mean curvature satisfying $\|H\| \geq c > 0$ (c a constant), then $R \leq \frac{1}{c}$. So, in particular, if z is defined in all $\mathbf{R}^2$ and $\|H\|$ is constant, then $H = 0$. Ten years later this problem was extended and solved for the case of a map $f: \mathbf{R}^m \to \mathbf{R}$ by Chern [2, Corollary] and, independently, by Flanders [5].

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2. Notations and main results

Let (M^m, g), (N^n, h) be Riemannian manifolds, $f\colon M \to N$ a smooth map, and Γ_f the graph of f, reading

$$\begin{aligned} \Gamma_f\colon M &\to (M \times N, g \times h) \\ x &\to \quad (x, f(x)). \end{aligned}$$

Note that as personal preference we are taking Γ_f to be an embedding instead of a set. Hence, we have on M two metrics, viz. g and the one induced by Γ_f, $\Gamma_f^*(g \times h) = g + f^*h$, which makes $\Gamma_f\colon (M, g + f^*h) \to (M \times N, g \times h)$, an isometric immersion. Let ∇ and ∇^* denote the Levi-Civita connections on (M, g) and $(M, g + f^*h)$, respectively. In general, we will use connections supplied with an asterisk $(*)$ to indicate that we are taking on M the metric $g + f^*h$.

Let V be the normal bundle of Γ_f in $\Gamma_f^{-1}(T(M \times N)) = T(M) \times f^{-1}T(N)$, and $\nabla^* d\Gamma_f \in C^\infty(\odot^2 T^*(M) \otimes V)$ the second fundamental form of the immersion Γ_f. The mean curvature of Γ_f is

$$H_{\Gamma_f} = 1/m \operatorname{Trace}_{(g+f^*h)}(\nabla^* d\Gamma_f) \in C^\infty(V).$$

Hence, Γ_f is a minimal immersion if and only if $H_{\Gamma_f} = 0$, and Γ_f has parallel mean curvature if and only if $\nabla^\perp H_{\Gamma_f} = 0$, where $\nabla^\perp$ denotes the covariant derivative in the normal bundle V.

We recall that the Cheeger constant of an oriented Riemannian manifold (M, g) is defined by (here we abusively adopt the same definition as for the compact case)

$$\mathfrak{h}(M) = \inf_D \frac{A(\partial D)}{V(D)},$$

where D ranges over all open submanifolds of M with compact closure in M and smooth boundary (see e.g. [1]). This constant is zero, if, for example, M is compact (without boundary) or (M, g) is a simple Riemannian manifold, that is, where there exists a diffeomorphism $\phi\colon (M, g) \to (\mathbf{R}^m, \langle , \rangle)$ onto $\mathbf{R}^m$ such that $\lambda g \le \phi^*\langle , \rangle \le \mu g$ for some positive constants λ, μ. Now we state our first main result:

Theorem 1. *If $\Gamma_f\colon (M, g+f^*h) \to (M \times N, g \times h)$ is an immersion with parallel mean curvature, then for each oriented compact domain $D \subset M$ we have the isoperimetric inequality*

$$c \le \frac{1}{m} \frac{A(\partial D)}{V(D)},$$

where $c = \|H_{\Gamma_f}\|_{g\times h}$ (c a constant) and where $V(D)$, $A(\partial D)$ are the volume of D resp. the area of ∂D, relative to the metric g.

In other words, if (M, g) is an oriented Riemannian manifold, then $\|H_{\Gamma_f}\|_{g\times h} \le 1/m\mathfrak{h}(M)$. In particular, if (M, g) has zero Cheeger constant, then Γ_f is in fact a minimal submanifold of $M \times N$.

On the other hand, if (M, g) is a complete, simply connected m-dimensional Riemannian manifold with sectional curvature bounded from above by $-K$, where K is a positive constant, then $\mathfrak{h}(M) \geq (m-1)\sqrt{K}$, with equality in the case where (M, g) is the m-hyperbolic space (see [10] and [1, pp. 95–96]). So, in the latter cases, we cannot expect a graph of a map $f\colon M \to N$ with parallel mean curvature to be minimal. In fact, the condition of vanishing Cheeger constant on (M, g) is fundamental for a graph with parallel mean curvature to be minimal, as we show with the following example.

Theorem 2. *Consider the* 2*-dimensional hyperbolic space* (H^2, g) *of constant sectional curvature* -1; *that is,* H^2 *is the unit open disk of* $\mathbf{R}^2$ *with center at the origin and* g *is the Riemannian metric given by*

$$g = \frac{4|dx|^2}{(1-|x|^2)^2}.$$

Then the function $f\colon H^2 \to \mathbf{R}$ *given by*

$$f(x) = \int_0^{r(x)} \sqrt{\tfrac{1}{2}(\cosh(r)-1)}\, dr = \frac{2}{\sqrt{1-|x|^2}} - 2,$$

where $r(x) = \log(\frac{1+|x|}{1-|x|})$ *is the distance function from the origin in* H^2, *is smooth on all* H^2, *and* $\Gamma_f \subset H^2 \times \mathbf{R}$ *has constant* (*and thus parallel*) *mean curvature with* $\|H_{\Gamma_f}\| = \frac{1}{2}$.

The proof of Theorem 2 is a straightforward calculation (for details, see [8]).

Remark 1. In the author's Ph.D. thesis [8], whereof this paper forms a part, for each constant $c \in (1-m, m-1)$ an example of a smooth map $f\colon H^m \to \mathbf{R}$ is given such that $\Gamma_f \subset H^m \times \mathbf{R}$ has constant mean curvature with $\|H_{\Gamma_f}\| = \frac{|c|}{m}$. This map is a function of the distance function from the origin, and for $c = 0$ it is the null function. This means that we could not find a nontrivial minimal graph of $H^m \times \mathbf{R}$. In [8] a sort of Bernstein theorem for such graphs is conjectured.

Henceforth, for each point $x \in M$, $(e_i)_{1\leq i\leq m}$ denotes an orthonormal basis of (T_xM, g), $(u_\alpha)_{1\leq\alpha\leq m}$ an orthonormal basis of $(T_xM, g + f^*h)$, $X_1, \dots, X_m$ a local orthonormal frame of (M, g) around a given point $x_0 \in M$, satisfying $\nabla X_i(x_0) = 0$, $\tilde{g}_{ij} = \langle X_i, X_j\rangle_{g+f^*h} = \delta_{ij} + \langle df(X_i), df(X_j)\rangle_h \forall i$, $j \in \{1, \dots, m\}$, and $(\tilde{g}^{ij})_{1\leq i,j\leq m}$ denotes the inverse of the matrix $(\tilde{g}_{ij})$. Let $(\,,)^{\perp}$ and $(\,,)^{\perp}$ denote the orthogonal projections of $T(M) \times f^{-1}T(N)$ on V resp. $d\Gamma_f(T(M))$, relative to the metric $g \times h$. Throughout this paper the ranges of indices are as follows: $1 \leq i, j, k, p, \alpha \leq m$.

Let $\nabla df \in C^\infty(\odot^2 T^*(M) \otimes f^{-1}T(N))$ be the second fundamental form of the map f (M with the metric g), and $\nabla^{f^{-1}}$, $\nabla^{\Gamma_f^{-1}}$ denote the connections on $f^{-1}T(N)$ and $\Gamma_f^{-1}(T(M \times N))$, respectively.

First we formulate the following useful lemmas:

Lemma 1. *For each* $X, Y \in C^{\infty}(T(M))$ *we have*

(i) $\nabla^{*} d\Gamma_f(X, Y) = (0, \nabla df(X, Y))^{\perp}$,

(ii) $mH_{\Gamma_f} = (-Z, W - df(Z)) = (0, W)^{\perp}$,
where $W = \mathrm{Trace}_{(g+f^*h)}(\nabla df)$ *and* Z *is the smooth vector field of* M *given at each point of* M *by* $Z = \sum_{i,j} \tilde{g}^{ij} \langle W, df(e_i)\rangle_h e_j$,

(iii) $m\nabla_X^{\Gamma_f^{-1}} H_{\Gamma_f} = (0, \nabla_X^{f^{-1}} W - \nabla df(X, Z)) - (\nabla_X Z, df(\nabla_X Z))$.

Proof. We first note that, if $X, Y \in C^{\infty}(T(M))$ and $U \in C^{\infty}(f^{-1}T(N))$, then $(X, U) \in C^{\infty}(\Gamma_f^{-1} T(M \times N))$ and

$$\nabla_Y^{\Gamma_f^{-1}}(X, U) = (\nabla_Y X, \nabla_Y^{f^{-1}} U).$$

Hence, using standard computations, we get

$$\nabla^{*} d\Gamma_f(X, Y) = (0, \nabla df(X, Y)) + (\nabla_X Y - \nabla_X^{*} Y,\ df(\nabla_X Y - \nabla_X^{*} Y)).$$

Since $\nabla^{*} d\Gamma_f(X, Y) \in C^{\infty}(V)$, we obtain (i). So we have

$$\begin{aligned} mH_{\Gamma_f} &= \sum_{i,j} \tilde{g}^{ij} \nabla^{*} d\Gamma_f(e_i, e_j) = \left(0, \sum_{i,j} \tilde{g}^{ij} \nabla df(e_i, e_j)\right)^{\perp} \\ &= (0, \mathrm{Trace}_{(g+f^*h)}(\nabla df))^{\perp} = (0, W)^{\perp} = (0, W) - (0, W)^{\perp} \\ &= (0, W) - \sum_{\alpha} \langle (0, W), (u_\alpha, df(u_\alpha))\rangle_{g\times h} (u_\alpha, df(u_\alpha)) \\ &= (0, W) - \sum_{\alpha} \langle W, df(u_\alpha)\rangle_h (u_\alpha, df(u_\alpha)). \end{aligned}$$

As $Z = \sum_{i,j} \tilde{g}^{ij} \langle W, df(e_i)\rangle_h e_j = \sum_\alpha \langle W, df(u_\alpha)\rangle_h u_\alpha$, we obtain $mH_{\Gamma_f} = (0, W) - (Z, df(Z))$. Finally, (iii) follows from (ii). □

Lemma 2. *Let* $x \in M$, $X \in T_xM$, *and* $v \in T_{f(x)}N$. *Then,* $(X, 0), (0, v) \in T_xM \times T_{f(x)}N$, *and*

(i) $v = 0$ *if and only if* $(0, v)^{\perp} = 0$,

(ii) $(X, 0) \in V_x$ *if and only if* $X = 0$.

Proof. We have

$$(0, v)^{\perp} = (0, v) - (0, v)^{\top} = \left(-\sum_\alpha \langle v, df(u_\alpha)\rangle_h u_\alpha, v - \sum_\alpha \langle v, df(u_\alpha)\rangle_h df(u_\alpha)\right).$$

Hence, $(0, v)^{\perp} = (0, 0)$ implies $\langle v, df(u_\alpha)\rangle_h = 0\ \forall\alpha$, and so $(0, 0) = (0, v)^{\perp} = (0, v)$. This proves (i). Now, if $(X, 0) \in V_x$, then, $\forall Y \in T_xM, \langle (X, 0), (Y, df(Y))\rangle_{g\times h} = 0$. So, $\langle X, Y\rangle_g = 0$. Hence, $X = 0$. □

Remark 2. From the above lemmas we can easily deduce that Γ_f is a totally geodesic submanifold of $M \times N$ if and only if $f: (M, g) \to (N, h)$ is a totally geodesic map, and that Γ_f is minimal if and only if $W = \mathrm{Trace}_{(g+f^*h)}(\nabla df) = 0$ if and only if $f: (M, g + f^*h) \to (N, h)$ is a harmonic map in the sense of [4] (see also [3]).

Lemma 3. *If Γ_f has parallel mean curvature, the following equality holds*:

$$m\langle \nabla^{\Gamma_f^{-1}} H_{\Gamma_f}, d\Gamma_f \rangle = -\operatorname{div}_g(Z),$$

where Z is as in Lemma 1, *and $\langle , \rangle$ is the Hilbert-Schmidt inner product.*

Proof. Since Γ_f has parallel mean curvature, from Lemma 1 (iii) we have $\forall X \in C^\infty(T(M))$,

$$0 = m\nabla_X^\perp H_{\Gamma_f} = m(\nabla_X^{\Gamma_f^{-1}} H_{\Gamma_f})^\perp = (0, \nabla_X^{f^{-1}} W - \nabla df(X, Z))^\perp,$$

whence, from Lemma 2 (i), $\nabla_X^{f^{-1}} W = \nabla df(X, Z)$, and so $m\nabla_X^{\Gamma_f^{-1}} H_{\Gamma_f} = -(\nabla_X Z, df(\nabla_X Z))$.

Let us now fix a point $x_0 \in M$. Then, $Z = \sum_{i,j} \tilde{g}^{ij} \langle W, df(X_i)\rangle_h X_j$ in a neighborhood of x_0. At the point x_0, since $\nabla X_i(x_0) = 0$, we have

$$\begin{aligned}\nabla_{X_i} Z &= \sum_{k,p} \nabla_{X_i}(\tilde{g}^{kp}\langle W, df(X_k)\rangle_h X_p) \\ &= \sum_{k,p} d(\tilde{g}^{kp}\langle W, df(X_k)\rangle_h)(X_i) X_p,\end{aligned}$$

so $\forall i, j$

$$\begin{aligned}&\langle (\nabla_{X_i} Z, df(\nabla_{X_i} Z)),\ (X_j, df(X_j))\rangle_{g\times h} \\ &\quad = \sum_{k,p} \langle d(\tilde{g}^{kp}\langle W, df(X_k)\rangle_h)(X_i)(X_p, df(X_p)),\ (X_j, df(X_j))\rangle_{g\times h} \\ &\quad = \sum_{k,p} \tilde{g}_{pj} d(\tilde{g}^{kp}\langle W, df(X_k)\rangle_h)(X_i),\end{aligned}$$

and, therefore,

$$\begin{aligned} m\langle \nabla^{\Gamma_f^{-1}} H_{\Gamma_f}, d\Gamma_f\rangle(x_0) &= \sum_{i,j} \tilde{g}^{ij}\langle m\nabla_{X_i}^{\Gamma_f^{-1}} H_{\Gamma_f}, d\Gamma_f(X_j)\rangle_{g\times h} \\ &= -\sum_{i,j} \tilde{g}^{ij}\langle (\nabla_{X_i} Z, df(\nabla_{X_i} Z)),\ (X_j, df(X_j))\rangle_{g\times h} \\ &= -\sum_{i,j,k,p} \tilde{g}^{ij}\tilde{g}_{pj} d(\tilde{g}^{kp}\langle W, df(X_k)\rangle_h)(X_i) \\ &= -\sum_{i,j,k,p} \delta_{ip} d(\tilde{g}^{kp}\langle W, df(X_k)\rangle_h)(X_i) \\ &= -\sum_{i,k} d(\tilde{g}^{ki}\langle W, df(X_k)\rangle_h)_{x_0}(X_i).\end{aligned}$$

Since $\sum_k \tilde{g}^{ki}\langle W, df(X_k)\rangle_h = \langle Z, X_i\rangle g$ in a neighborhood of x_0, we get

$$\begin{aligned} m\langle \nabla^{\Gamma_f^{-1}} H_{\Gamma_f}, d\Gamma_f\rangle(x_0) &= -\sum_i d(\langle Z, X_i\rangle_g)_{x_0}(X_i) \\ &= -\sum_i \langle \nabla_{X_i} Z, X_i\rangle_g(x_0) \\ &= -\operatorname{div}_g(Z)(x_0). \quad \square \end{aligned}$$

Proof of Theorem 1. Recall the following formula (see [4]) for a map ϕ: $(P, g') \to (\widetilde{P}, \tilde{g})$ between Riemannian manifolds,

$$\operatorname{div}_{g'}(\langle d\phi(\cdot), \tau_\phi\rangle_{\tilde{g}}) = \|\tau_\phi\|^2_{\tilde{g}} + \langle d\phi, \nabla^{\phi^{-1}}\tau_\phi\rangle,$$

where τ_ϕ is the tension field of ϕ. Then, since Γ_f: $(M, g + f^*h) \to (M \times N, g \times h)$ is an isometric immersion, the tension field of Γ_f is mH_{Γ_f}, and $H_{\Gamma_f} \perp d\Gamma_f(T(M))$, so

$$0 = \operatorname{div}_{(g+f^*h)}(\langle d\Gamma_f(\cdot), mH_{\Gamma_f}\rangle_{g\times h}) = m^2\|H_{\Gamma_f}\|^2_{g\times h} + m\langle \nabla^{\Gamma_f^{-1}} H_{\Gamma_f}, d\Gamma_f\rangle.$$

Hence

$$\langle \nabla^{\Gamma_f^{-1}} H_{\Gamma_f}, d\Gamma_f\rangle = -m\|H_{\Gamma_f}\|^2_{g\times h}. \tag{1}$$

Therefore, from Lemma 3 we obtain

$$m^2c^2 = \operatorname{div}_g(Z).$$

Now let $D \subset M$ be an oriented compact domain and dV_g, dA_g the respective volume elements of D, ∂D, relative to the metric g. Then by applying Stokes' theorem, we get

$$m^2c^2V(D) = \int_D m^2c^2\,dV_g = \int_D \operatorname{div}_g(Z)\,dV_g = \int_{\partial D}\langle Z, \nu\rangle_g\,dA_g,$$

where ν is the outward unit normal of ∂D. From the Schwarz inequality $|\langle Z, \nu\rangle_g| \le \|Z\|_g\|\nu\|_g = \|Z\|_g$, and Lemma 1 (ii), we obtain $mc = m\|H_{\Gamma_f}\|_{g\times h} = \|(-Z, W - df(Z))\|_{g\times h} \ge \|Z\|_g$. Hence, we finally obtain

$$m^2c^2V(D) \le \int_{\partial D}|\langle Z, \nu\rangle_g|\,dA_g \le \int_{\partial D} mc\,dA_g = mc\,A(\partial D). \quad \square$$

Remark 3. As a consequence of Theorem 1, Remark 2, and Hopf's maximum principle (see e.g. [1]), if M is an oriented compact manifold, $N = \mathbf{R}^n$, and Γ_f has parallel mean curvature, then f is a constant map.

3. Some particular cases

The key to obtaining the inequality in Theorem 1 was Lemma 3, which allowed us (using Eq. (1)) to write the square of the mean curvature of Γ_f as the divergence of a vector field. If the graph Γ_f is a hypersurface of $M \times N$, that is, N is of dimension one, then we can derive some similar conclusions about the mean curvature of Γ_f, without the assumption that Γ_f has parallel mean curvature, which was required in Lemma 3. In fact, we have the following:

Proposition 1. *Assume N is oriented and of dimension one.*

(a) *If $D \subset M$ is an oriented compact domain of M, then*

$$\min_{x\in\overline{D}} \|H_{\Gamma_f}\|_{g\times h} \leq \frac{1}{m}\frac{A(\partial D)}{V(D)},$$

where the volumes of D and of ∂D are taken relative to the metric g. In particular, if (M, g) has Cheeger constant equal to zero, then $\inf_M \|H_{\Gamma_f}\|_{g\times h} = 0$.

(b) *If (M, g) is a connected, oriented, complete Riemannian manifold, and*

$$\lim_{R\to+\infty} \frac{1}{R}\int_{B_R(x_0)} \left(\frac{\|df\|}{\sqrt{1+\|df\|^2}}\right) dV_g = 0$$

for some x_0, where $B_R(x_0)$ is the geodesic ball with center x_0 and radius R, and $\|df\|$ is the Hilbert-Schmidt norm of df, then there exists an $x \in M$ such that $H_x = 0$. Moreover, if $\langle H_{\Gamma_f}, \nu\rangle_{g\times h}$ is contained in $[0, +\infty)$ or in $(-\infty, 0]$, where ν is a unit normal to Γ_f along all Γ_f, then $H_{\Gamma_f} \equiv 0$.

Proof. The computations in this proof are essentially the same as in [5], so here we give only a sketch of the proof. As noted above, we wish to write the mean curvature of Γ_f as a divergence of some bounded vector field. In fact, we have the equalities

$$\begin{aligned} m\langle H_{\Gamma_f}, \nu\rangle_{g\times h} &= \frac{1}{m}\left\langle \tau_f - \frac{1}{w^2}\sum_{i,j}\langle df(X_i), Y\rangle_h \langle df(X_j), Y\rangle_h \nabla\, df(X_i, X_j), Y\right\rangle_h \\ &= \operatorname{div}_g\left(\frac{\nabla f}{w}\right), \end{aligned}$$

where Y is a unit vector field on all (N, h); $w = \sqrt{1+\|df\|^2}$; $\nabla f \in C^\infty(T(M))$ is given by $\langle \nabla f_x, u\rangle_g = \langle df_x(u), Y_x\rangle_h$ $\forall u \in T_xM$; $\nu = \frac{1}{w}(-\nabla f, Y)$ is a unit normal of Γ_f; and τ_f is the tension field of $f\colon (M, g) \to (N, h)$. Then (a) results from applying Stokes' theorem and from the fact that $\|\frac{\nabla f}{w}\|_g \leq 1$, and (b) results as an application of the extended Stokes' theorem of Gaffney-Yau (see [6], [9, Appendix]). □

The mean curvature of a graph Γ_f of an isometric immersion f is strongly related to the mean curvature of f. More generally, if f is a conformal map, then the mean curvature of Γ_f can be expressed in terms of the tension field of f, as we show in the following:

Proposition 2. *Let $f\colon (M, g) \to (N, h)$ be a (weakly) conformal map, that is, $f^*h = \lambda^2 g$, where $\lambda\colon M \to \mathbf{R}_0^+$ is a smooth map. Then,*

(a) $mH_{\Gamma_f} = (0, (1+\lambda^2)^{-1}\tau_f)^\perp$, *where τ_f is the tension field of f;*

(b) *Γ_f is a minimal submanifold of $(M\times N, g\times h)$, if and only if $f\colon (M,g) \to (N,h)$ is a harmonic map (and in this case, for $m\neq 2$, is a homothetic map)*;

(c) *if f is a homothetic map or $m=2$, then Γ_f has parallel mean curvature, if and only if Γ_f is minimal, if and only if $\nabla^{f^{-1}}((1+\lambda^2)^{-1}\tau_f)=0$;*

(d) *if $m\neq 2$ and Γ_f has parallel mean curvature, then*

$$\Delta((1+\lambda^2)^{-1}) = \frac{2m^2}{m-2}c^2$$

with $c=\|H_{\Gamma_f}\|_{g\times h}$ (a constant). Consequently,

(i) *if (M,g) is parabolic or if λ has a minimum on $M\sim\partial M$, then Γ_f is minimal*;

(ii) *if (M,g) is complete, connected, and oriented, and $m\geq 3$, then, for $V(M,g)<+\infty$, Γ_f is minimal, and for $V(M,g)=+\infty$, $(1+\lambda^2)^{-1}\notin L^p(M,g)$ $\forall p\in(1,+\infty)$.*

Proof. Since $f^*h=\lambda^2 g$, $\Gamma_f^*(g\times h) = g+f^*h = (1+\lambda^2)g=\mu^2 g$, where $\mu\colon M\to[1,+\infty)$ is a smooth map. Then it follows from standard calculations that

$$(2) \qquad mH_{\Gamma_f} = \mu^{-2}(0,\tau_f) + (m-2)\mu^{-2}(w, df(w)),$$

where $w=\nabla(\log\mu)$ is the gradient of log μ relative to the metric g. Hence,

$$mH_{\Gamma_f} = (mH_{\Gamma_f})^\perp = (0,\mu^{-2}\tau_f)^\perp,$$

and (b) is proved by applying Lemma 2 (i). If $m\neq 2$ and Γ_f is minimal, we have from Eq. (2) $w=0$, that is, f is homothetic (see also [4]). If f is a homothetic map or $m=2$, we obtain

$$mH_{\Gamma_f} = \mu^{-2}(0,\tau_f).$$

In particular, $\tau_f\perp_g df(T(M))$ and, $\forall X\in C^\infty(T(M))$,

$$m\nabla_X^{\Gamma_f^{-1}} H_{\Gamma_f} = (0,\nabla_X^{f^{-1}}(\mu^{-2}\tau_f)), \qquad m\nabla_X^\perp H_{\Gamma_f} = (0,\nabla_X^{f^{-1}}(\mu^{-2}\tau_f))^\perp.$$

Hence, using Lemma 2 (i), Eq. (1), and the last two equations, we get $m\nabla^\perp H_{\Gamma_f}=0$ iff $\nabla^{f^{-1}}(\mu^{-2}\tau_f)=0$ iff $m\nabla^{\Gamma_f^{-1}}H_{\Gamma_f}=0$ iff $H_{\Gamma_f}=0$, and we have proved (c). In order to obtain (d) we must first prove the formula

$$\langle\tau_f, df(\cdot)\rangle_h = \frac{2-m}{2}d\lambda^2.$$

We have at a point x_0 (see §2 for notations), $\forall i,j,k$,

$$\begin{aligned}\langle\nabla df(X_i,X_j), df(X_k)\rangle_h &= \langle\nabla_{X_i}^{f^{-1}}(df(X_j)), df(X_k)\rangle_h \\ &= \delta_{jk}d\lambda^2(X_i) - \langle df(X_j), \nabla df(X_i,X_k)\rangle_h.\end{aligned}$$

Performing a cyclic permutation on the indices i, j, k, and adding in a convenient way the resulting expressions of these equations, we get at x_0

$$\langle \nabla df(X_i, X_j), df(X_k)\rangle_h = \tfrac{1}{2}\{\delta_{jk} d\lambda^2(X_i) - \delta_{ij} d\lambda^2(X_k) + \delta_{ki} d\lambda^2(X_j)\}.$$

Putting $i = j$ and tracing in the index i, we get $\langle \tau_f, df(X_k)\rangle_h = \frac{1}{2}(2-m)\, d\lambda^2(X_k)\ \forall k$, and we have proved the desired formula. Now, supposing that Γ_f has parallel mean curvature, then from Lemma 3 and Eq. (1) we have, at a point x_0, $m^2c^2 = \sum_i d(\sum_k \tilde{g}^{ki}\langle W, df(X_k)\rangle_h)_{x_0}(X_i)$, where $W = \sum_{i,j} \tilde{g}^{ij} \nabla df(X_i, X_j)$ in a neighborhood of x_0. Since $\tilde{g}_{ij} = \mu^2\delta_{ij}$, $W = \sum_i \mu^{-2}\nabla df(X_i, X_i) = \mu^{-2}\tau_f$, and $\langle W, df(X_k)\rangle_h = \mu^{-2}\langle \tau_f, df(X_k)\rangle_h$. Thus, at x_0 we have

$$\begin{aligned}
m^2c^2 &= \sum_i d(\sum_k \delta_{ik}\mu^{-2}\langle \mu^{-2}\tau_f, df(X_k)\rangle_h)(X_i)\\
&= \sum_i d(\mu^{-4}\langle \tau_f, df(X_i)\rangle_h)(X_i)\\
&= \sum_i d(\tfrac{1}{2}(2-m)\mu^{-4}\, d\lambda^2(X_i))(X_i)\\
&= \frac{m-2}{2}\sum_i \nabla d(\mu^{-2})_{x_0}(X_i, X_i)\\
&= \frac{m-2}{2}\Delta(\mu^{-2})(x_0).
\end{aligned}$$

Hence, the equation in (d) holds with $0 < \mu^{-2} \le 1$. So

$$\text{for } m \ge 3, \ \begin{cases} \Delta(\mu^{-2}) \ge 0 \\ \mu^{-2} \le 1 \end{cases} \quad \text{and for } m = 1, \ \begin{cases} \Delta(\mu^{-2}) \le 0 \\ \mu^{-2} \ge 0 \end{cases}.$$

Hence, if (M, g) is parabolic, μ must be constant, and therefore $0 = \Delta(\mu^{-2}) = \frac{2m^2}{m-2}c^2$, i.e. Γ_f is minimal. Now (d) (i) follows from Hopf's maximum principle. For $m \ge 3$, we get $\mu^{-2}\Delta(\mu^{-2}) \ge 0$. So, from Theorem 3 of [9], we have either $\int_M (\mu^{-2})^p\, dV_g = +\infty\ \forall p \in (0,1)\cup(1, +\infty)$ or μ is constant. Thus, if the volume of (M, g) is finite, we conclude that μ is constant, i.e. Γ_f is minimal; and if it is infinite, we easily deduce, from the negation, that $\mu^{-2} \notin L^p(M, g)\ \forall p \in (0,1)\cup(1, +\infty)$. □

Remark 4. As we have seen in Remark 2, for $\Gamma_f\colon (M, g+f^*h) \to (M\times N, g\times h)$ to be a minimal immersion is in general not equivalent to $f\colon (M, g) \to (N, h)$ being harmonic. In Proposition 2 we saw that this equivalence holds for a conformal map. We also remark that if $f\colon (M, g) \to (N, h)$ is a Riemannian submersion, then the equivalence also holds. In fact, using computations similar to those used in the proof of Proposition 2, we get the equality $mH_{\Gamma_f} = (0, \tau_f)^{\perp}$ (cf. [8]). Moreover, we can also show that Γ_f has parallel mean curvature if and only if $\|\tau_f\|_h$ is constant and if and only if the fibers of f have a constant mean curvature whose norm is the same for all fibers.

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