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A P-ADIC ANALOGUE OF THE GAUSS-BONNET THEOREM FOR CERTAIN MUMFORD CURVES

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ABSTRACT. If K is a local field, L a quadratic extension, Γ a Schottky group co-compact in $\mathrm{PGL}_2(K)$ then the quotient $L - K/\Gamma$ corresponds to the L-points of a Mumford curve. In this paper we calculate $\int_{L-K/\Gamma} d\mathscr{M}$, where $\mathscr{M}$ is an $\mathrm{PGL}_2(K)$ invariant measure on $L - K$, in terms of the genus of the corresponding curve.

In this paper we will give a p-adic analogue of the Gauss-Bonnet theorem in the following case. Let X be a compact Riemann surface with genus $g \geq 2$. Then there is a uniformization of X by the hyperbolic upper half plane $\mathscr{H}$. That is, $X \approx \mathscr{H}/\Gamma$ where $\mathscr{H} = \{x + iy \in \mathbf{C} | y > 0\}$ with the usual hyperbolic structure and Γ is a Fuchsian subgroup of $\mathrm{SL}_2(\mathbf{R})$. The area form $dx\,dy/y^2$ is invariant under $\mathrm{SL}_2(\mathbf{R})$ and since the curvature of $\mathscr{H}$ is constant -1 the Gauss-Bonnet formula says:

$$\int_{X=\mathscr{H}/\Gamma} dx\,dy/y^2 = -2\pi\chi(X) = 2\pi(2g-2).$$

1. MUMFORD CURVES

(a) *Notation.* Throughout this paper K will denote a field complete with respect to a discrete valuation $v\colon K^* \to \mathbf{Z}$ with the convention $v(0) = +\infty$. We assume that $\mathrm{char}(K) \neq 2$. Let $\mathscr{O}_K = \mathscr{O}$ denote the ring of integers. That is, $\mathscr{O} = \{x \in K | v(x) \geq 0\}$. Take π to be a uniformizer in $\mathscr{O}$, so $v(\pi) = 1$. Then π will generate the maximal ideal of $\mathscr{O}$ and $k = \mathscr{O}/\pi\mathscr{O}$ is the residue field. We assume throughout that k is finite of order q. We take the normalized absolute value on K, i.e. $|x| = q^{-v(x)}$ for $x \in K$.

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As usual $\mathbf{P}^1_K$ will denote the projective line over K and the rational points $\mathbf{P}^1(K) = K \cup \infty$ can be given by homogeneous coordinates so:

$\mathbf{P}^1(K) = \{[x, y] | x, y \in K\}/[x, y] \sim \lambda[x, y]; \lambda \in K^*$ (Identify $a \in K$ with $[a, 1]$.)

Let $G = \mathrm{PGL}_2(K) = \mathrm{GL}_2(K)/K^*$, where we embed $K^* \to \mathrm{GL}_2(K)$ by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Then G acts on $\mathbf{P}^1(K)$ by linear fractional transformations, i.e. If $g \in G$ is represented by a matrix $g \sim \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and $x \in \mathbf{P}^1(K)$ then:

$$g(x) = \frac{ax + b}{cx + d}.$$

(b) *Schottky Groups.*

Definition 1. A subgroup $\Gamma \subset G$ is called a Schottky group if 1) Γ is finitely generated and 2) every element $\gamma \in \Gamma$ is conjugate in G to a matrix of the form $\left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $q \in \pi\mathscr{O}$.

Any $\gamma \in G$ which satisfies 2 is called hyperbolic. It is easy to see that any hyperbolic γ has two distinct fixed points in $\mathbf{P}^1(K)$. A theorem of Ihara's (See [2] or [8, p. 82]) implies that every Schottky group is free.

Definition 2. Given a Schottky group Γ we define $\Sigma \subset \mathbf{P}^1(K)$ to be the closure of the set of fixed points of elements $\gamma \in \Gamma$, $\gamma \neq \mathrm{id}$.

The group Γ will then act properly discontinuously on $\Omega(K)$, the complement of Σ in $\mathbf{P}^1(K)$.

Theorem 1 (Mumford). *The quotient $X_\Gamma(K) = \Gamma \backslash \Omega(K)$ has the structure of the K-rational points of an algebraic curve X_Γ defined over K with genus $g(X_\Gamma) = g = \mathrm{rank}(\Gamma)$.*

(For a proof see Mumford's paper [5] or the survey article [4]. For a proof in the category of rigid analytic spaces see [1].)

Curves that are given by such quotients are called Mumford curves and can be thought of as p-adic Riemann surfaces.

(c) *The case of co-compact* Γ. In the case Γ is co-compact in G, $\Sigma = \mathbf{P}^1(K)$ (For a proof that co-compactness implies $\Sigma = \mathbf{P}^1(K)$ see [4, p. 315 Theorem 6.7] and [8, p. 84]) so $\Omega(K)$ has no points at all. In this case the corresponding Mumford curve is defined over K but has no K-rational points. If L is any finite extension of K, we can view $K \subset L$ and $X_\Gamma(L) = \Gamma \backslash \Omega(L)$ where

$$\Omega(L) = \mathbf{P}^1(L) - \mathbf{P}^1(K) = L - K.$$

So in this case the region of discontinuity of Γ in $\mathbf{P}^1(L)$ very closely resembles the classical upper half plane. In fact if L is a quadratic extension of K so that $L = K(\delta)$ with $\delta^2 \in K$ we see that $\Omega(L) = \{x + y\delta | x, y \in K, y \neq 0\}$.

For the rest of this paper we will restrict ourselves to the quadratic case and use coordinates for $\Omega(L)$ as above. Now, if we let dx and dy be forms on K which induce Haar measure normalized so that $\int_{\mathscr{O}_K} dx = \int_{\mathscr{O}_K} dy = 1$ then we have the form $dx\,dy/|y|^2$ on $\Omega(L)$. Just as the corresponding form in the classical case is $\mathrm{PSL}_2(\mathbf{R})$ invariant on $\mathscr{H}$, it is easy to see that $dx\,dy/|y|^2$ is $\mathrm{PGL}_2(K) = G$ invariant on $\Omega(L)$.

(d) *The Main Result.* We now can state the main result of this paper as:

Theorem 2. *Let* Γ *be a Schottky group co-compact in* $\mathrm{PGL}_2(K)$ *and let* L *be a quadratic extension of* K, $L = K(\delta) = \{x + y\delta | y \neq 0\}$, *then*:

$$\int_{X_\Gamma(L)} dx\,dy/|y|^2 = \begin{cases} (2g-2)/q & \text{if } L/K \text{ is unramified}, \\ (q+1)(g-1)/q^2 & \text{if } L/K \text{ is ramified}. \end{cases}$$

We will give two proofs of this result, the first depending on some measure theory and a similar result of Serre and the second by actually constructing a fundamental domain for Γ in $\Omega(L)$ over which this form can be integrated.

2. The first proof of the Theorem 2

As noted above this proof will be measure theoretic in nature and will rely strongly on the following:

Theorem 3 (Serre). *If* μ *is the unique Haar measure on* G *with* $\mu(\mathrm{PGL}_2(\mathscr{O})) = (q-1)/2$ *then* $\mu(\Gamma\backslash G) = g - 1$.

Proof. [8, p. 82] also see [7], (As cited in the references, this statement is slightly different as it involves the analogous result for SL_2.)

The idea now is to express $\Omega(L)$ as G/T for some subgroup T. This is analogous to writing $\mathscr{H} = \mathrm{PSL}_2(\mathbf{R})/\mathrm{PSO}_2(\mathbf{R})$.

Let

$\mathscr{A} \subset \mathrm{GL}_2(K)$ be the group of matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$,

$\mathscr{B} \subset \mathrm{GL}_2(K)$ be the group of matrices of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$,

$\mathscr{T} \subset \mathrm{GL}_2(K)$ be the group of matrices of the form $\begin{pmatrix} c & dD \\ d & c \end{pmatrix}$,

where $D = \delta^2$.

Denote by A, B, T the images of these groups in G. Notice that T is the stabilizer of δ under the action of G on $\Omega(L)$. Since this action is transitive we identify $\Omega(L) \approx G/T$.

Proposition 1. $G = ABT$.

Proof. Notice first that these groups have pairwise intersection equal to the identity. What we have to show is that any element $g \in G$ can be uniquely represented $g = abt$, with $a \in A$, $b \in B$, $t \in T$. Since we know that

$\Omega(L) = G/T$ it is enough to show that $\Omega(L) \approx AB$ but this is clear since we can identify $x + y\delta$ with ab where;

$$a = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & x/y \\ 0 & 1 \end{pmatrix}.$$

Notice that $A \approx K^*$, $B \approx K$ with coordinates a, b respectively so the forms $da/|a|$ and db which give the regular Haar measures on K^* and K also give Haar measures on A and B. (Recall the usual normalization; $\int_{\mathscr{O}^*} da = (q-1)/q$ and $\int_{\mathscr{O}} db = 1$.) (Compare this decomposition to that of $\mathrm{SL}_2(\mathbf{R})$ [3].)

Proposition 2. *If we identify* $\Omega(L)$ *with* AB, *with coordinates as above then the form* $dx\,dy/|y|^2 \longleftrightarrow da\,db/|a|$.

Proof. This is just a simple change of coordinates, $x \leftrightarrow a$ and $y \leftrightarrow ba$, so

$$dx\,dy/|y|^2 = |a|\,da\,db/|a|^2 = da\,db/|a|.$$

Proposition 3. *Under appropriate normalization of the measure on* T *induced by* dt, *we have*:

$$d\mu = da\,db\,dt/|a|.$$

Proof. Since $dx\,dy/|y|^2$ is a G invariant measure on Ω and $\Omega = G/T$, then (up to normalization) $d\mu = (dx\,dy/|y|^2)\,dt$. Now use Proposition 2.

Proposition 4. *If* L/K *is unramified, then* $T \subset U$, $U = \mathrm{PGL}_2(\mathscr{O})$.

Proof. In the unramified case we can take δ to be a unit in $\mathscr{O}$. We must show that any element of T can be represented by a matrix with entries in $\mathscr{O}$ and determinant in $\mathscr{O}^*$. Let $t \in T$; then clearly one can find a matrix representing t with entries in $\mathscr{O}$. In coordinates this amounts to giving c, $d \in \mathscr{O}$ with

$$\begin{pmatrix} c & dD \\ d & c \end{pmatrix} \qquad \text{representing } t.$$

We can also assume that either c or d is in $\mathscr{O}^*$. To see that $c^2 - d^2D$ is a unit in $\mathscr{O}$ notice that this is $N_{L/K}(c + d\delta)$ but $c + d\delta$ is a unit in $\mathscr{O}_L$ since if π is a uniformizer in $\mathscr{O}$ it is one in $\mathscr{O}_L$ since L/K is unramified and $\pi \nmid c + d\delta$.

We are now in a position to prove Theorem 2 in the unramified case:

Theorem 2a. *If* L/K *is unramified then*

$$\int_{X_\Gamma(L)} dx\,dy/|y|^2 = (2g-2)/q.$$

Proof. $X_\Gamma(L) = \Gamma\backslash\Omega = \Gamma\backslash G/T$. So

$$\begin{aligned}\int_{X_\Gamma(L)} dx\,dy/|y|^2 &= \int_{\Gamma\backslash G/T} da\,db\,dt/|a| \\ &= \int_{\Gamma\backslash G} d\mu \Big/ \int_T dt \\ &= (g-1) \Big/ \int_T dt \qquad \text{(By Theorem 3)}.\end{aligned}$$

So the theorem will be proved if we can show that $\int_T dt = q/2$. To this end we calculate:

$$\int_U d\mu = \int_U da\, db\, dt/|a| = (q-1)/2.$$

It is easy to show that $U = (A \cap U)(B \cap U)(T \cap U)$ so

$$(q-1)/2 = \left(\int_{A\cap U} da/|a|\right)\left(\int_{B\cap U} db\right)\left(\int_{T\cap U} dt\right).$$

Recall that the coordinates a for A and b for B give isomorphisms $A \approx K^*$, $B \approx K$ and we have $A \cap U \approx \mathscr{O}^*$ and $B \cap U \approx \mathscr{O}$. By the proposition above we see $T \cap U = T$ so

$$\begin{aligned}(q-1)/2 &= \left(\int_{\mathscr{O}^*} da/|a|\right)\left(\int_{\mathscr{O}} db\right)\left(\int_T dt\right) \\ &= ((q-1)/q)(1)\left(\int_T dt\right).\end{aligned}$$

Thus $\int_T dt = q/2$.

The ramified case is a bit harder since it is no longer true that $T \subset U$. First we notice that if L/K is ramified then $L = K(\delta)$ where $D = \delta^2$ is a uniformizer for the maximal ideal of $\mathscr{O}$ so we can assume $D = \pi$. Let $\mathscr{I}$ be the subgroup of $\mathrm{GL}_2(\mathscr{O})$ consisting of matrices of the form $\left(\begin{smallmatrix} a & b\pi \\ c & d\end{smallmatrix}\right)$ and let I be its image in U. We have:

Proposition 5. *(a) I is of index $q+1$ in U.*

(b) $I = (A \cap I)(B \cap I)(T \cap I)$.

(c) $I \cap T$ is of index 2 in T.

Proof. For (a) see [8, p. 77]. (b) is easy to check.

(c) Notice that any $t \in T$ is equivalent either to the identity or $\left(\begin{smallmatrix} 0 & \pi \\ 1 & 0\end{smallmatrix}\right)$ in $T/I \cap T$.

Now we can prove

Theorem 2b. *If L/K is ramified then*

$$\int_{X_\Gamma(L)} dx\, dy/|y|^2 = (q+1)(g-1)/q^2.$$

Proof. As above we have $\int_{X_\Gamma(L)} dx\, dy/|y|^2 = (g-1)/\int_T dt$. We calculate

$$\int_I d\mu = \left(\int_{I\cap U} da/|a|\right)\left(\int_{I\cap U} db\right)\left(\int_{I\cap U} dt\right).$$

By (a) above we get $\int_I d\mu = (q-1)/2(q+1)$. Now, $I \cap A \approx \mathscr{O}^*$ and $I \cap B \approx \pi\mathscr{O}$ so

$$\begin{aligned}(q-1)/2(q+1) &= \left(\int_{\mathscr{O}^*} da/|a|\right)\left(\int_{\pi\mathscr{O}} db\right)\left(\frac{1}{2}\int_T dt\right) \\ &= \frac{1}{2}((q-1)/q)(1/q)\left(\int_T dt\right).\end{aligned}$$

Thus $\int_T dt = q^2/(q+1)$ and the theorem follows.

3. The second proof of Theorem 2

(a) *The tree* $\mathcal{T}_K$. The second proof of the main theorem will depend on an explicit construction of a fundamental domain for the action of Γ on $\Omega(L)$. In order to do this we need to introduce the tree $\mathcal{T}_K$ associated to the field K. (c.f. Serre [8] or Mumford [5]).

We define the graph $\mathcal{T}_K = \mathcal{T}$ in the following way: The vertices of $\mathcal{T}$,

$$\mathrm{Vert}(\mathcal{T}) = \{M \subset K \oplus K | M \text{ is a two dimensional } \mathcal{O}-\text{lattice}\}/\sim,$$

where $M \sim N$ if there exists $\lambda \in K^*$ such that $\lambda M = N$. Given a lattice M we denote its class by [M] and the corresponding vertex by v_M. Two vertices v_M and v_N are connected by an edge if there are $M' \in [M]$ and $N' \in [N]$ such that $M' \subset N'$ and $N'/M' \approx \mathcal{O}/\pi = k$. We shall list the following facts about $\mathcal{T}$, their proofs can be found in [5] or [8].

Theorem 4. (1) *$\mathcal{T}$ is a homogeneous tree of degree $q+1$.*

(2) *The $q+1$ vertices adjacent to a given vertex can be identified with the points of $\mathbf{P}^1(k)$.*

(3) *The 1/2-lines in $\mathcal{T}$ based at a fixed vertex v (That is, the collection of infinite paths in $\mathcal{T}$ based at v.) correspond to points in $\mathbf{P}^1(K)$.*

(4) *The action of G on $K \oplus K$ induces an action of G on $\mathrm{Vert}(\mathcal{T})$ which is isometric where distance is measured by the number of edges between two vertices. Thus we can think of G as acting on the tree $\mathcal{T}$.*

(5) *The action of G on $\mathrm{Vert}(\mathcal{T})$ is transitive and the stabilizer of $v_{\mathcal{O}\oplus\mathcal{O}} = v_0$ is $U = \mathrm{PGL}_2(\mathcal{O})$. Thus $\mathrm{Vert}\,\mathcal{T} = G/U$.*

(6) *If Γ is a Schottky group then Γ acts freely on $\mathcal{T}$, that is it leaves on edges or vertices fixed. If Γ is co-compact in G then the quotient $\Gamma\backslash\mathcal{T}$ is a finite graph.*

By 3 we can think of $\mathbf{P}^1(K)$ as being the "boundary" of this infinite graph. Choosing a base point for the 1/2-lines corresponds to picking coordinates for the projective line at infinity; in fact if we let v_0 be the vertex of $\mathcal{T}$ corresponding to $\mathcal{O}\oplus\mathcal{O}$ then we can label the vertices distance n away from v_0 by the following scheme. (Distance is measured without backtracking.)

$$\{\text{Vertices distance } n \text{ from } v_0\} \longleftrightarrow \{v_L\}$$

$$L \in \Big\{\Big[\sum a_i\pi^i,\ 1\Big]\mathcal{O} + [\pi^n,\ 0]\mathcal{O} \text{ or } [0,\ \pi^n]\mathcal{O} + \Big[1,\ \sum a_i\pi^i\Big]\mathcal{O};$$

$$a_i \text{ representing cosets of } \mathcal{O}/\pi\Big\}$$

(See [6] for a further explanation of the labeling.)

If we notice that any point in $\mathbf{P}^1(K)$ can be is represented uniquely by a homogeneous coordinate of the form $[a, 1]$ or $[1, \pi a]$ with $a \in \mathscr{O}$ then the above remarks say that we can label vertices distance n from v_0 by these coordinates " mod π^n ".

Now we will assume Γ is a co-compact Schottky group and we have the formula

Proposition 6. *If* $h =^{\#}$ *vertices of* $\Gamma\backslash\mathscr{T}$ *and* $e =^{\#}$ *geometric edges of* $\Gamma\backslash\mathscr{T}$ *then* $g - 1 = e - h$. (*Recall* $g = \text{rank}(\Gamma)$).

Proof. Both sides of this equation calculate $-\chi(\Gamma\backslash\mathscr{T}) = -$ Euler characteristic of this graph.

Definition 3. (1) If $x \in \mathbf{P}^1(K)$ then we denote by $\mathbf{x}$ the $1/2$ line in $\mathscr{T}$ corresponding to x with base point v_0.

(2) If $v \in \text{Vert}(\mathscr{T})$ then

$$R^{-1}(v) = \{x \in \mathbf{P}^1(K) | \mathbf{x} \text{ passes through } v\}.$$

It is easy to see that $R^{-1}(v)$ is a disc of radius q^{-d}, where d is the distance from v to v_0, and center at the point whose homogeneous coordinate labels v.

(b) *Trees under field extensions.* Suppose L/K is a quadratic extension; then we have the corresponding trees $\mathscr{T}_K$ and $\mathscr{T}_L$. There is an inclusion $\text{Vert}(\mathscr{T}_K) \rightarrowtail \text{Vert}(\mathscr{T}_L)$ given by the map $M \rightarrowtail M \otimes \mathscr{O}_L$ where M is an $\mathscr{O}_K$-lattice.

In the case L/K is unramified then $\mathscr{T}_L$ is a homogeneous tree of degree $q^2 + 1$ and we can identify $\mathscr{T}_K$ in $\mathscr{T}_L$ as a subtree. In any case we have $\text{Im}(\mathscr{T}_K) = \bigcup_{x \in \mathbf{P}^1(K) \subset \mathbf{P}^1(L)} \mathbf{x}$.

In the ramified case we have $\mathscr{T}_K \approx \mathscr{T}_L$ and the image of $\mathscr{T}_K$ is no longer a homogeneous tree but rather looks like $\mathscr{T}_K$ with each edge subdivided into two.

(c) *Fundamental Domains.* As we stated above, if Γ is a Schottky group co-compact in G then $\Gamma\backslash\mathscr{T}_K$ is a finite graph. Define a subtree J of $\mathscr{T}_K$ as follows. Begin with v_0 and an adjacent edge, continue to add edges and vertices so that the result is connected and no edge or vertex is equivalent to any other under the action of Γ. The resulting subtree is J. It is the lift of a maximal subtree of the quotient $\Gamma\backslash\mathscr{T}_K$.

Proposition 7. J *is a fundamental domain for the action of* Γ *on* $\mathscr{T}_K$.

Proof. [8, p. 25].

Now we can describe the fundamental domain for Γ in $\Omega = L - K$ as follows. We consider the image of $\mathscr{T}_K$ in $\mathscr{T}_L$ and let $\mathscr{J}$ be the image of J in the tree $\mathscr{T}_L$.

Definition 4. If v is any vertex of $\mathcal{T}_L$ we define $R^{-1}(v, L) = \{x \in \mathbf{P}^1(L) | \mathbf{x}$ goes through v and each vertex further from v_0 than v in $\mathbf{x}$ is in $\mathrm{Vert}(\mathcal{T}_L) - \mathrm{Im}[\mathrm{Vert}(\mathcal{T}_K)]\}$.

In other words, if $v \in \mathrm{Vert}(\mathcal{T}_L) - \mathrm{Im}[\mathrm{Vert}(\mathcal{T}_K)]$ then $R^{-1}(v, L) = R^{-1}(v)$ and $v \in \mathrm{Im}[\mathrm{Vert}(\mathcal{T}_K)]$ then $R^{-1}(v, L)$ is the set of all x such that the 1/2-line $\mathbf{x}$ when traced from v_0 leaves the subtree $\mathrm{Im}(\mathcal{T}_K)$ after passing through v.

Proposition 8. *The set* $\Lambda = \bigcup_{v \in \mathcal{J}} R^{-1}(v, L)$ *is a fundamental domain for the action of* Γ *on* Ω.

Proof. [4, p. 316].

(d) *Volume calculations.*

Proposition 9. *Assume* L/K *is unramified then for any* $v \in \mathrm{Im}[\mathrm{Vert}(\mathcal{T}_K)]$ *we have*

$$\int_{R^{-1}(v, L)} dx\, dy/|y|^2 = (q-1)/q^2.$$

Proof. Since G acts transitively on $\mathcal{T}_K$, it is enough to prove this proposition for $v = v_0$ since the form we are integrating is G invariant and $R^{-1}(g(v), L) = g(R^{-1}(v, L))$. Also notice that $R^{-1}(v_0, L) = \bigcup R^{-1}(v)$ where the union is taken over those v adjacent to v_0 and in $\mathrm{Vert}(\mathcal{T}_L) - \mathrm{Im}[\mathrm{Vert}(\mathcal{T}_K)]$. In terms of our coordinates these are vertices adjacent to v_0 that can be labeled by homogeneous coordinates of the form $[a + b\delta, 1]$ where a, $b \in k$ and $b \neq 0$. $R^{-1}(v)$ then is $\{x + y\delta | \mathrm{ord}_\pi[(x + y\delta) - (a + b\delta)] \geq 1\}$. That is, $R^{-1}(v)$ is the set of all points congruent to $a + b\delta \bmod \pi$. Now it is easy to see

$$\int_{R^{-1}(v)} dx\, dy/|y|^2 = \int_{R^{-1}(v)} dx\, dy = 1/q^2.$$

Since this calculation is valid for every vertex adjacent to v_0 and in $\mathrm{Vert}(\mathcal{T}_L) - \mathrm{Im}[\mathrm{Vert}(\mathcal{T}_K)]$ and the corresponding $R^{-1}(v)$'s are disjoint we get,

$$\int_{R^{-1}(v_0, L)} dx\, dy/|y|^2 = \sum \int_{R^{-1}(v)} dx\, dy = q(q-1)/q^2 = (q-1)/q.$$

The second equality following since we are summing over $(q^2 - 1) - (q - 1) = q(q-1)$ vertices.

We can now prove our main theorem in the unramified case, for

$$\int_{X_\Gamma(L)} dx\, dy/|y|^2 = \sum_{v \in \mathcal{J}} \int_{R^{-1}(v, L)} dx\, dy/|y|^2 = h(q-1)/q = (2g-2)/q.$$

($h(q-1) = 2g - 2$ follows from Proposition 6 and $e = h(q+1)/2$.)

In the ramified case we take the same approach; however things are a bit more complicated as not every vertex in $\mathrm{Im}(\mathcal{T}_K)$ comes from a vertex of $\mathcal{T}_K$. We write $\mathcal{V} = \mathrm{Vert}[\mathrm{Im}(\mathcal{T}_K)] - \mathrm{Im}[\mathrm{Vert}(\mathcal{T}_K)]$

Proposition 10. *If* $v \in \mathrm{Im}[\mathrm{Vert}(\mathcal{T}_K)]$ *then* $R^{-1}(v, L) = \varnothing$.

If $v \in \mathcal{V}$ *then*

$$\int_{R^{-1}(v,L)} dx\,dy/|y|^2 = (q-1)/q^2.$$

Proof. First notice that if $v \in \mathrm{Im}[\mathrm{Vert}(\mathcal{T}_K)]$ then each edge adjacent to v stays in the subtree $\mathrm{Im}(\mathcal{T}_K)$; thus $R^{-1}(v, L) = \varnothing$. Now for $v \in \mathcal{V}$ we have $q-1$ vertices adjacent to v but not in $\mathrm{Im}(\mathcal{T}_K)$. We have, $R^{-1}(v, L) = \bigcup R^{-1}(v_i)$ where the union is over these vertices. Again we notice that since G acts transitively on the edges of $\mathcal{T}_K$ it is sufficient to carry out the calculation of this integral for any vertex in $\mathcal{V}$. It is easiest to take v to be the vertex adjacent to v_0 labeled by $[0, 1]$, then $R^{-1}(v, L) = \bigcup_{v_i \text{ labeled by } [b\delta, 1], b\in k, b\neq 0} R^{-1}(v_i)$ and

$$\int_{R^{-1}(v,L)} dx\,dy/|y|^2 = \sum_{v_i \text{ labeled by } [b\delta,1], b\in k, b\neq 0} \int_{R^{-1}(v_i)} dx\,dy/|y^2|.$$

For each such v_i we get $\int_{R^{-1}(v_i)} dx\,dy/|y|^2 = \int_{R^{-1}(v_i)} dx\,dy = 1/q^2$ so

$$\int_{R^{-1}(v,L)} dx\,dy/|y|^2 = (q-1)/q^2.$$

Now to complete the proof of the main theorem we calculate:

$$\begin{aligned}\int_{X_\Gamma(L)} dx\,dy/|y|^2 &= \sum_{v\in\mathcal{J}} \int_{R^{-1}(v,L)} dx\,dy/|y|^2 \\ &= \sum_{v\in\mathcal{J}\cap\mathcal{V}} \int_{R^{-1}(v,L)} dx\,dy/|y|^2 \\ &= \sum_{v\in\mathcal{J}\cap\mathcal{V}} (q-1)/q^2 \\ &= [h(q+1)/2][(q-1)/q^2],\end{aligned}$$

since # vertices in $\mathcal{J} \cap \mathcal{V} = $ # edges of $\Gamma\backslash\mathcal{T}_K = h(q+1)/2$. Finally we have

$$\int_{X_\Gamma(L)} dx\,dy/|y|^2 = (g-1)(q+1)/q^2.$$

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